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A Constructive Proof of Luft's Theorem in Case Genus Two.

Robert John Kramer

Louisiana State University and Agricultural & Mechanical College

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A CONSTRUCTIVE PROOF OF LUFT'S THEOREM IN CASE GENUS TWO

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A CONSTRUCTIVE PROOF OF LUFT'S THEOREM
IN CASE GENUS TWO

A Dissertation

Submitted to the Graduate Faculty of the
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in

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by

Robert John Kramer
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I dedicate this on December nineteenth to my parents, Mr. and Mrs. H. John Kramer of Virginia Beach, Virginia.

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ABSTRACT

In Kirby's problem list [R. Kirby: Problems in Low Dimensional Manifold Theory. Proc. Symp. Pure Math. 32, (1978), p.28] is "Problem 2.4: (Birman) Let α be the obvious homomorphism $\eta_g \xrightarrow{\alpha} \text{Aut}(\pi_1(N_g))$ where η_g is the group of isotopy classes of orientation preserving homeomorphisms of N_g . Is kernel (α) finitely generated?" Here N_g denotes the 3-dimensional orientable handlebody of genus g . See [J. Birman: Braids, Links, and Mapping Class Groups. Ann. of Math. Studies No. 82, PUP(1975), p. 220]. In [E. Luft: Actions of the Homeotopy Group of an Orientable 3-Dimensional Handlebody. Math. Ann. 234 (1978), Corollary 2.3] Luft proves that kernel (α) is generated by Dehn twists along properly embedded 2-cells in N_g . In [J. Birman: Private communication. Aug. 6, 1979] it was suggested that a geometric proof of Luft's result be found since Luft's proof was algebraic in nature. The author gives a constructive geometric proof of Luft's Theorem in the case of a handlebody of genus two.

Chapter 1: Preliminaries

Section 1.1: The PL category

For a general source on the PL category see [R-S].
All spaces and all maps in this paper are PL.

Some notation follows. If S is a subspace of X , then the closure of S in X is denoted by $\text{Cl}(S)$. If $f: A \rightarrow B$ is a function, then the image of f , $\{f(a) | a \in A\}$, is denoted by $\text{Im } f$.

Section 1.2: Manifolds

If M is a n -manifold, the interior of M is $\text{Int } M = \{x \in M | x \text{ has a neighborhood homeomorphic to } \mathbb{R}^n\}$, and the boundary of M is $\text{Bd } M = M \setminus \text{Int } M$. An n -cell ($(n-1)$ -sphere) is a space homeomorphic to an n -simplex (the boundary of an n -simplex). A 1-cell is called an arc. If L is a submanifold of M , then L is properly embedded in M if $L \cap \text{Bd } M = \text{Bd } L$.

Section 1.3: Isotopy

If $H: X \times I \rightarrow Y$ is a map, define $H_t: X \rightarrow Y$ for each $t \in I$ by $H_t(x) = H(x, t)$ for each $x \in X$. A map, $H: X \times I \rightarrow X$, is an isotopy if each $H_t: X \rightarrow X$ is a homeomorphism of X onto X . An isotopy, $H: X \times I \rightarrow X$, is a mod S isotopy of X if S is a subset of S and $H_t(s) = s$ for each $t \in I$, $s \in S$. An isotopy, $H: X \times I \rightarrow X$,

is invariant on S if S is a subset of X and $H_t(S) = S$ for each $t \in I$. For proofs of the following two Theorems see [R-S].

Theorem 1.3.1: If M is a manifold with compact boundary, then any isotopy of BdM extends to one of M .

Theorem 1.3.2: Let B be an n -cell. If $h: B \rightarrow B$ is a homeomorphism of B onto B and $h|_{BdB}$ is the identity, then h is mod BdB isotopic to the identity.

Section 1.4: Orientation

For a discussion of orientation see [R-S]. For a proof of the following Theorem see [G].

Theorem 1.4.1: If M is an n -cell or an n -sphere, then any orientation-preserving homeomorphism of M onto itself is isotopic to the identity.

Section 1.5: Regular neighborhoods

For a discussion of regular neighborhoods see [R-S]. If K is a compact polyhedron in a manifold M , then a choice of a regular neighborhood for K in M will always mean a second derived neighborhood of K with respect to a triangulation of M which contains K as a subcomplex. A proof of the following Theorem can be found in [R-S].

Theorem 1.5.1: Suppose N_1 and N_2 are regular neighborhoods of a compact polyhedron K in a manifold M . Then there is a mod K isotopy, $H: M \times I \rightarrow M$, such that $H_1(N_1) = N_2$.

Section 1.6: Two-sided embeddings

A compact $(n-1)$ -manifold F is 2-sided in an n -manifold M if there is an embedding $h: F \times [-1, 1] \rightarrow M$ with $h(x, 0) = x$ for each $x \in F$ and $h(F \times [-1, 1]) \cap \text{Bd } M = h((\text{Bd } F) \times [-1, 1])$. For a proof of the following Theorem see [H].

Theorem 1.6.1: If F is a compact $(n-1)$ -manifold properly embedded in an n -manifold M and if image $(i_*: H_1(F; \mathbb{Z}_2) \rightarrow H_1(M; \mathbb{Z}_2)) = 0$, then F is 2-sided in M .

For F a 2-sided $(n-1)$ -manifold in an n -manifold M and $h: F \times [-1, 1] \rightarrow M$ an embedding as above, the n -manifold $M \setminus h(F \times (-1, 1))$ is called the result of cutting M along F .

Section 1.7: Two-cells with n holes

If D is a 2-cell and D_1, \dots, D_n ($n \geq 0$) is a collection of mutually disjoint 2-cells lying in $\text{Int } D$, then the space $D \setminus \text{Int}(D_1 \cup \dots \cup D_n)$ is called a 2-cell with n holes, and will be denoted by $D(n)$. It is well-known (see [M]) that a 2-cell with m holes is homeomorphic to a 2-cell with n holes if and only if $m = n$. A 2-cell

with 1 hole is called an annulus. A properly embedded arc c in $D(n)$ is called a spanning arc if $Bd c$ intersects two components of $Bd D(n)$. The result of cutting a 2-cell with n holes along a spanning arc is a 2-cell with $n-1$ holes. The result of cutting a 2-cell n holes along a properly embedded arc which is not a spanning arc is the disjoint union of a 2-cell with n' holes for some n' , $0 \leq n' \leq n$ and a 2-cell with $n-n'$ holes.

Section 1.8: Handlebodies

If N is an orientable 3-manifold containing a collection v_1, \dots, v_g of mutually disjoint 2-cells properly embedded in N such that the result of cutting N along $v = v_1 \cup \dots \cup v_g$ is a 3-cell, then N is called a handlebody (of genus g), and v is called a system of meridian disks (for N). If $P: v \times [-1, 1] \rightarrow N$ is a 2-sided embedding of v in N and $g > 0$, then the surface $(Bd N) \setminus (v \times (-1, 1))$ is a 2-cell with $2g-1$ holes and its boundary is $P((Bd v) \times \{-1, 1\})$. The following two Theorems follow from elementary "cut and paste" arguments.

Theorem 1.8.1: If N is a handlebody and v, v' are systems of meridian disks for N and $Bd v = Bd v'$, then there is a mod $Bd v$ isotopy, $H: N \times I \rightarrow N$, such that $H_1(v) = v'$.

Theorem 1.8.2: If N is a handlebody of genus g and w_1, \dots, w_g is a collection of mutually disjoint 2-cells properly embedded in N , then $w = w_1 \cup \dots \cup w_g$ is a system disks for N provided $(\text{Bd } N) \setminus \text{Bd } w$ is connected.

Section 1.9: General position

If F is a 2-manifold, then a system of curves in F is a properly embedded compact 1-submanifold of F . If k, ℓ are systems of curves in a 2-manifold, F , then k and ℓ are in general position if $k \cap \ell$ is a finite subset of $\text{Int } F$ and for each $p \in k \cap \ell$ there is an open neighborhood U in F and a homeomorphism $h: U \rightarrow \mathbb{R}^2$ onto \mathbb{R}^2 such that $h(U \cap k) = \mathbb{R} \times \{0\}$ and $h(U \cap \ell) = \{0\} \times \mathbb{R}$. If N is a handlebody, k is a 1-subcomplex of N , and $P: v \times [-1, 1] \rightarrow N$ is a 2-sided embedding of a system of meridian disks for N , then k is in general position with P if $k \cap v$ is finite and $k \cap \text{Im } P = P((k \cap v) \times [-1, 1])$.

Chapter 2: Luft's Theorem

If N is a handlebody and D is a properly embedded 2-cell in N , $P: D \times [-1,1] \rightarrow N$ is a 2-sided embedding of D in N , and $h: N \rightarrow N$ is a homeomorphism such that for each $x \in N \setminus P(D \times (-1,1))$ $h(x) = x$, then h is called a simple twist of N . It is easy to see that simple twists are homotopic to the identity.

If N is a handlebody and for each $i=1, \dots, n$ $h_i: N \rightarrow N$ is a simple twist of N or a homeomorphism of N isotopic to the identity, then the composition $h_n \circ \dots \circ h_1$ is called a twist of N .

A proof of the following Theorem will be included at the end of this chapter.

Theorem 2.1 [L,1]: Let N be a handlebody of genus two. Suppose $h: N \rightarrow N$ is an orientation-preserving homeomorphism which is homotopic to the identity. Then h is a twist of N .

If N is a handlebody, v is a system of meridian disks for N , and k is a system of curves in $Bd N$ in general position with respect to $Bd v$, then k is said to be cyclically reduced with respect to v if there exists a 2-sided embedding of v in N , $P: v \times [-1,1] \rightarrow N$, k is in general position with P , and each arc component c of $k \setminus P(v \times (-1,1))$ intersects two boundary components of $(Bd N) \setminus P(v \times (-1,1))$.

The following Lemma will be proved in a later chapter.

Lemma 2.2: Let N be a handlebody of genus two and let v be a system of meridian disks for N . Suppose k is a system of curves in BdN and no component of k is contractible in N . Then there is a twist T of N such that $T(k)$ is cyclically reduced with respect to v .

If N is a handlebody, $v = v_1 \cup \dots \cup v_g$ is a system of meridian disks for N , and $k = k_1 \cup \dots \cup k_g$ is a system of curves in BdN in general position with Bdv such that $k_i \cap v_j = \emptyset$ if $i \neq j$ and $k_i \cap v_i$ is exactly one point for each i , then k is said to be a system of curves in BdN which is conjugate to v .

The following four Lemmas will be proved in later chapters.

Lemma 2.3: If N is a handlebody, v is a system of meridian disks for N , and k is a system of curves in BdN which is conjugate to v , then no component of k is contractible in N .

Lemma 2.4: Let N be a handlebody, let v be a system of meridian disks for N , and let k be a system of curves in BdN which is conjugate to v . Suppose $h: N \rightarrow N$ is a homeomorphism which is homotopic to the identity and $h(k)$ is cyclically reduced with respect to v . Then $h(k)$ is conjugate to v .

Lemma 2.5: Let N be a handlebody of genus two, let v be a system of meridian disks for N , and let k be a system of curves in BdN which is conjugate to v . Suppose w is a system of meridian disks for N and k is conjugate to w . Then there is an isotopy $H: N \times I \rightarrow N$ such that $H_1(v) = w$.

Lemma 2.6: [L,1;p.285], [L,2]: Let N be a handlebody and let v be a system of meridian disks for N . Suppose $h: N \rightarrow N$ is an orientation-preserving homeomorphism which is homotopic to the identity and $h(v) = v$. Then h is a twist of N .

Proof of Theorem 2.1: Let N be a handlebody of genus two and suppose $h: N \rightarrow N$ is an orientation-preserving homeomorphism which is homotopic to the identity. Choose a system of meridian disks $v = v_1 \cup v_2$ for N . Choose a system $k = k_1 \cup k_2$ of curves in BdN which is conjugate to v . Now k_1 and k_2 are non-contractible by Lemma 2.3. Since h is a homeomorphism $h(k_1)$ and $h(k_2)$ are non-contractible in N . By Lemma 2.2 there is a twist T of N such that $T \circ h(k)$ is cyclically reduced with respect to v . By Lemma 2.4 $T \circ h(k)$ is conjugate to v . Since $T \circ h(k)$ is also conjugate to $T \circ h(v)$, by Lemma 2.5 there is an isotopy $H: N \times I \rightarrow N$ such that $H_1 \circ T \circ h(v) = v$. Now by Lemma 2.6 $H_1 \circ T \circ h$ is a twist T' of N . Then $h = T^{-1} \circ (H_1)^{-1} \circ T'$ and hence h is a twist of N . Thus the Theorem is proved. \square

Chapter 3: Homotopy

Section 3.1: The core of a handlebody

Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Associated with P there is a strong deformation retract Γ which will be constructed as follows. For each $i = 1, \dots, g$ choose a point $y_i \in \text{Int } v_i$ and let $y = \{y_1, \dots, y_g\}$. Let B denote the 3-cell $N \setminus P(v \times (-1, 1))$. Note that $P(y \times \{-1, 1\})$ is a collection of $2g$ points in $\text{Bd } B$. Let $h: B \rightarrow [-1, 1] \times [-1, 1] \times [-1, 1] = [-1, 1]^3$ be a homeomorphism of B onto the convex 3-cell $[-1, 1]^3$. Note that $h(P(y \times \{-1, 1\}))$ is a collection of $2g$ points in $\text{Bd}([-1, 1]^3)$. Let C denote the cone of $h(P(y \times \{-1, 1\}))$ in $[-1, 1]^3$ with vertex the point $(0, 0, 0)$ in $[-1, 1]^3$. Denote $h^{-1}(0, 0, 0) \in B$ by x_0 . Let $\Gamma = h^{-1}(C) \cup P(y \times [-1, 1])$. For each i let Γ_i denote the 1-sphere in Γ which contains the point $P(y_i, 0)$. Then Γ is the wedge of the collection $\Gamma_1, \dots, \Gamma_g$ and Γ has wedge point x_0 . Now choose a strong deformation retract $R: N \times I \rightarrow N$ of N onto Γ such that $R(P(v_i \times \{s\}) \times I) \subset P(v_i \times \{s\})$ for each $s \in [-1, 1]$ and such that $R(B \times I) \subset B$. Then Γ is called a core of N associated with P and R is called a retract associated with Γ .

Section 3.2: Cyclic words

Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N , let Γ be a core of N associated with P , and let R be a retract associated with Γ .

Let S denote the unit circle $\{\exp(it) | t \in \mathbb{R}\}$ in the complex numbers. For each i choose a homeomorphism $f_i: (S, \{1\}) \rightarrow (\Gamma_i, \{x_0\})$ such that $f_i(\exp(\frac{2}{3}\pi i)) = P(y_i, -1)$ and $f_i(\exp(\frac{4}{3}\pi i)) = P(y_i, 1)$. It is well-known that the fundamental group $\pi_1(\Gamma, x_0)$ is a free group with basis $\{[f_1], \dots, [f_g]\}$ [M, p. 125].

Suppose that k is a simple closed curve in $Bd N$ which is in general position with P . Choose a homeomorphism $f: S \rightarrow k$ of S onto k such that $f(1) \notin \text{Im } P$. Let $n = \text{card}(k \cap v)$. A finite sequence (a_1, \dots, a_n) in the set $\{[f_1], \dots, [f_g], [f_1]^{-1}, \dots, [f_g]^{-1}\}$ is defined which records the encounters and the direction of the encounters of k with v as follows. Define $e: I \rightarrow Bd N$ by $e(t) = f(\exp 2\pi it)$. Then $e^{-1}(Bd v) = \{t_1 < \dots < t_n\} \subset \text{Int}(I)$. For each $j = 1, \dots, n$ let $s_j = \min(e^{-1}(P(v \times \{-1, 1\})) \cap [t_j, 1])$. Then $e(s_j) \in P(v_{i(j)} \times \{\delta(j)\})$ for some $i(j) = 1, \dots, g$ and $\delta(j) = \pm 1$. Let $a_j = [f_{i(j)}]^{\delta(j)}$. The sequence (a_1, \dots, a_n) is called a cyclic word associated with k .

If it is the case that $R_1 \circ f(1) = x_0$, then the homotopy class $[R_1 \circ f] = a_1 \cdots a_n \in \pi_1(\Gamma, x_0)$.

Section 3.3: The proof of Lemma 2.3

Proof of Lemma 2.3: Let N be a handlebody of genus g , let v be a system of meridian disks for N , and let k be a system of curves in BdN which is conjugate to v . If $g=0$, then $k=\emptyset$ and there is nothing to prove. So suppose $g>0$. Let k_i be one of the components of $k=k_1 \cup \cdots \cup k_g$. Let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N such that k_i is in general position with P . Choose a homeomorphism $g: S \rightarrow k_i$ of S onto k_i such that $g(1) \notin \text{Im } P$, $g(\exp(\frac{2}{3}\pi i)) \in P(v_i \times \{-1\})$, and $g(\exp(\frac{4}{3}\pi i)) \in P(v_i \times \{1\})$. Then a cyclic word associated with k_i is $([f_i])$. Choose a mod $\text{Im } P$ isotopy $J: N \times I \rightarrow N$ such that $R_1 \circ J_1 \circ g(1) = x_0$. Then the homotopy class $[R_1 \circ J_1 \circ g] = [f_i]$ in $\pi_1(\Gamma, x_0)$. Therefore k_i is not contractible in N . It follows that no component of k is contractible. Hence the Lemma is proved. \square

Section 3.4: Free groups

Let F be a free group with basis $B = \{b_1, \dots, b_g\}$ $g>0$. Let $B^* = B \cup \{b_1^{-1}, \dots, b_g^{-1}\}$. A word $a_1 \cdots a_n \in F$ such that each $a_i \in B^*$ is said to be reduced if $a_i a_{i+1} \neq 1$ for each $i=1, \dots, n-1$ and cyclically reduced if it is reduced and $a_n a_1 \neq 1$. Each element of F

is equal to a unique reduced word. See [L-S, p.3].

Lemma 3.4.1: Let F be a free group with basis $B = \{b_1, \dots, b_g\}$ $g > 0$. If $a_1 \dots a_n$ is a reduced word, then the element of F defined by the product $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1}$ is not an element of the set B for any cyclically reduced word $c_1 \dots c_m$ with $m > 1$.

Proof of Lemma 3.4.1: Let F be a free group with basis $B = \{b_1, \dots, b_g\}$ $g > 0$. Let $a_1 \dots a_n$ be a reduced word and let $c_1 \dots c_m$ be a cyclically reduced word with $m > 1$. If $n = 0$, then $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1} = c_1 \dots c_m$. Since $c_1 \dots c_m$ is a reduced word of length $m > 1$, $c_1 \dots c_m \notin B$. Let $k \geq 0$. As an induction hypothesis assume that $a_1 \dots a_k c_1 \dots c_m a_k^{-1} \dots a_1^{-1} \notin B$ for any reduced word $a_1 \dots a_k$ and cyclically reduced word $c_1 \dots c_m$ with $m > 1$. Let $n = k + 1$, let $a_1 \dots a_n$ be a reduced word, and let $c_1 \dots c_m$ be a cyclically reduced word with $m > 1$. Suppose $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1} \in B$. Since $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1}$ is a word with $2n + m$ terms and $2n + m > 1$, there must be some cancellation. Since $a_1 \dots a_n$ and $c_1 \dots c_m$ are reduced, either $a_n c_1 = 1$ or $c_m a_n^{-1} = 1$. Suppose $a_n c_1 = 1$. Then $a_1 \dots a_{n-1} c_2 \dots c_m a_n^{-1} \dots a_1^{-1} \in B$. It will now be shown that $c_2 \dots c_m a_n^{-1}$ is a cyclically reduced word. If $c_m a_n^{-1} = 1$, then $c_m = a_n = c_1^{-1}$, which is impossible because $c_1 \dots c_m$ is cyclically reduced. Therefore $c_m a_n^{-1} \neq 1$. If $a_n^{-1} c_2 = 1$, then $c_2 = a_n = c_1^{-1}$, which is impossible because $c_1 \dots c_m$

is reduced. Therefore $c_2 \dots c_m a_n^{-1}$ is a cyclically reduced word. Since $a_1 \dots a_{n-1}$ is a reduced word of length k , by the induction hypothesis, $(a_1 \dots a_{n-1})(c_2 \dots c_m a_n^{-1})(a_{n-1}^{-1} \dots a_1^{-1}) \notin B$. Therefore $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1} \notin B$. Suppose on the other hand that $c_m a_n^{-1} = 1$. Then it can be proved in an analogous fashion that $a_n c_1 \dots c_{m-1}$ is a cyclically reduced word of length $m > 1$. By the induction hypothesis $(a_1 \dots a_{n-1})(a_n c_1 \dots c_{m-1})(a_{n-1}^{-1} \dots a_1^{-1}) \notin B$. Therefore the Lemma is proved. \square

Lemma 3.4.2: Let F be a free group with basis $B = \{b_1, \dots, b_g\}$ $g > 0$. Let $a_1 \dots a_n$ be a reduced word and let $b \in B^*$. If the element of F defined by the product $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1}$ is an element of the set B^* then $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1} = b$ in F .

Proof of Lemma 3.4.2: Let F be a free group with basis $B = \{b_1, \dots, b_g\}$ $g > 0$. Let $a_1 \dots a_n$ be a reduced word and let $b \in B^*$. Suppose $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1} \in B^*$. If $n = 0$, then $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1} = b$ and the Lemma is true. Let $k \geq 0$. As an inductive hypothesis assume that if $a_1 \dots a_k$ is a reduced word and $b \in B^*$, then $a_1 \dots a_k b a_k^{-1} \dots a_1^{-1} = b$. Let $n = k + 1$, let $a_1 \dots a_n$ be a reduced word, and let $b \in B^*$. Suppose $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1} \in B^*$. Since $a_1 \dots a_n$ is reduced and the number of terms in $a_1 \dots a_n b a_n^{-1} \dots a_1^{-1}$ is $2n + 1 > 1$, either $a_n b = 1$ or $b a_n^{-1} = 1$. Suppose $a_n b = 1$. Then $a_n^{-1} = b$. Therefore

$a_1 \cdots a_n b a_n^{-1} \cdots a_1^{-1} = a_1 \cdots a_{n-1} b a_{n-1}^{-1} \cdots a_1^{-1} \in B^*$. Since the length of the reduced word $a_1 \cdots a_{n-1}$ is k , by the induction hypothesis $a_1 \cdots a_{n-1} b a_{n-1}^{-1} \cdots a_1^{-1} = b$ and so $a_1 \cdots a_n b a_n^{-1} \cdots a_1^{-1} = b$. Suppose on the other hand that $b a_n^{-1} = 1$. Then $a_n = b$. Hence $a_1 \cdots a_{n-1} b a_{n-1}^{-1} \cdots a_1^{-1} \in B^*$. Therefore by the induction hypothesis $a_1 \cdots a_{n-1} b a_{n-1}^{-1} \cdots a_1^{-1} = b$. Thus the Lemma is proved. \square

Section 3.5: The proof of Lemma 2.4

Let N be a handlebody, let v be a system of meridian disks for N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . If k is a system of curves in $Bd N$ such that if c is an arc component of $k \setminus P(v \times (-1,1))$, then c intersects two distinct boundary components of $(Bd N) \setminus P(v \times (-1,1))$, then k is said to be cyclically reduced with respect to P .

Proof of Lemma 2.4: Let N be a handlebody of genus g , let v be a system of meridian disks for N , and let k be a system of curves in $Bd N$ such that k is conjugate to v . Suppose $h: N \rightarrow N$ is a homeomorphism which is homotopic to the identity and $h(k)$ is cyclically reduced with respect to v . Then $h(k)$ is cyclically reduced with respect to P for some 2-sided embedding of v in N , $P: v \times [-1,1] \rightarrow N$. There is a mod v isotopy of N , $H: N \times I \rightarrow N$, such that $H_1(k)$ is in general position with P .

If $g=0$, then there is nothing to prove. So suppose $g>0$. Let Γ be a core of N associated with P , let x_0 denote the vertex of Γ , and let R denote an associated retract. Since there is a mod $Bd N$ isotopy, $H': N \times I \rightarrow N$, such that $H'_1 \circ h(x_0) = x_0$, it can be assumed without loss of generality that $h(x_0) = x_0$. Let k_i be one of the components of $k = k_1 \cup \dots \cup k_g$. It will be shown that $h(k_i) \cap v_j = \emptyset$ if $i \neq j$ and $h(k_i) \cap v_i$ is a single point. It can then be concluded from the definition that $h(k)$ is conjugate to v . Let B denote the 3-cell $N \setminus P(v \times (-1, 1))$. Let $x_1 \in k_i \setminus \text{Im } P$. Let $g: [0, 1] \rightarrow B$ be an embedding such that $g(0) = x_0$ and $(\text{Im } g) \cap Bd B = \{x_1\}$. Let S denote the unit circle in the complex numbers. Let $f: S \rightarrow k_i$ be a homeomorphism of S onto k_i with $f(1) = x_1$, $f(\exp \frac{2}{3} \pi i) \in P(v_i \times \{-1\})$ and $f(\exp \frac{4}{3} \pi i) \in P(v_i \times \{1\})$. Define a map $\varphi: S \rightarrow N$ as follows. If $\exp(\frac{2}{3} \pi i t) \in S$ for some $t \in [0, 3]$, let

$$\varphi(\exp \frac{2}{3} \pi i t) = \begin{cases} g(t) & \text{if } t \in [0, 1] , \\ f(\exp(2\pi i(t-1))) & \text{if } t \in [1, 2] , \\ g(3-t) & \text{if } t \in [2, 3] . \end{cases}$$

Then $\varphi(1) = x_0$ and the homotopy class $[R_1 \circ \varphi] = [f_i]$ in $\pi_1(\Gamma, x_0)$ where $f_i: S \rightarrow N$ was defined in Section 3.2. Since there is an isotopy $J: N \times I \rightarrow N$ such that J is invariant on $h(k_i) \cup \{x_0\}$, $J_1 \circ h(x_1) \notin \text{Im } P$, and $J_1(\text{Im } \varphi)$ is in general position with P , it can be assumed without

loss of generality that $h(x_1) \notin \text{Im } P$ and $\text{Im } \varphi$ is in general position with P .

Since $h(k_i)$ is a simple closed curve in $\text{Bd } N$ which is in general position with P and $h \circ f: S \rightarrow h(k_i)$ is a homeomorphism of S onto $h(k_i)$ such that $h \circ f(1) = h(x_1) \notin \text{Im } P$, a cyclic word (c_1, \dots, c_m) associated with $h(k_i)$ can be defined as in Section 3.2 with respect to the homeomorphism $h \circ f$, where each $c_j \in \{[f_1], \dots, [f_g], [f_1]^{-1}, \dots, [f_g]^{-1}\}$. Moreover, the word $c_1 \dots c_m$ is a cyclically reduced word in $\pi_1(\Gamma, x_0)$ because $h(k_i)$ is cyclically reduced with respect to P . Since $h(\text{Im } g)$ is in general position with P , a finite sequence (d_1, \dots, d_r) in the set $\{[f_1], \dots, [f_g], [f_1]^{-1}, \dots, [f_g]^{-1}\}$ can be defined which records the encounters and the direction of the encounters of $h(\text{Im } g)$ with v which will have the property that the homotopy class $[R_1 \circ h \circ \varphi] = d_1 \dots d_r c_1 \dots c_m d_r^{-1} \dots d_1^{-1}$ in $\pi_1(\Gamma, x_0)$. Since h is homotopic to the identity, $[R_1 \circ h \circ \varphi] = [f_i]$ in $\pi_1(\Gamma, x_0)$. Let $a_1 \dots a_n$ be a reduced word in $\pi_1(\Gamma, x_0)$ with each $a_k \in \{[f_1], \dots, [f_g], [f_1]^{-1}, \dots, [f_g]^{-1}\}$ such that $a_1 \dots a_n = d_1 \dots d_r$ in $\pi_1(\Gamma, x_0)$. Since $a_1 \dots a_n c_1 \dots c_m a_n^{-1} \dots a_1^{-1} = [f_i]$, by Lemma 3.4.1 $m = 0$ or $m = 1$. Since m cannot equal zero, it follows that $m = 1$. Therefore $a_1 \dots a_n c_1 a_n^{-1} \dots a_1^{-1} = [f_i]$. Therefore $c_1 = a_n^{-1} \dots a_1^{-1} [f_i] a_1 \dots a_n$. Since $a_n^{-1} \dots a_1^{-1}$ is reduced, $[f_i] \in B^*$, and $c_1 \in B^*$, it follows from Lemma 3.4.2 that

$c_1 = [f_i]$. Now (c_1) is a cyclic word for $h(k_i)$. Hence $([f_i])$ is a cyclic word for $h(k_i)$. It follows that $h(k_i) \cap v_j = \emptyset$ if $i \neq j$ and $h(k_i) \cap v_i$ is a single point. Thus the Lemma is proved. \square

Section 3.6: Contractible simple closed curves

Lemma 3.6.1: Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose k is a simple closed curve in BdN in general position with P such that $k \cap v \neq \emptyset$ and k is contractible in N . Then at least two arc components of $k \setminus P(v \times (-1,1))$ are not spanning arcs in the 2-cell with $2g-1$ holes $(BdN) \setminus P(v \times (-1,1))$.

Proof of Lemma 3.6.1: Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose k is a simple closed curve in BdN in general position with P such that $k \cap v \neq \emptyset$ and k is contractible in N . Let S denote the unit circle in the complex numbers. Let $f: S \rightarrow k$ be a homeomorphism of S onto k with $f(1) \notin \text{Im } P$. Let Γ be a core of N associated with P , and let R be a retract associated with Γ , and let $\{[f_1], \dots, [f_g]\}$ be the basis for $\pi_1(\Gamma, x_0)$ defined in Section 3.2. Let (a_1, \dots, a_n) be a cyclic word associated

with k defined as in Section 3.2 with respect to the homeomorphism f . Then there is a mod $\text{Im } P$ isotopy of N , $H: N \times I \rightarrow N$, such that $R_1 \circ H_1 \circ f$ defines a map from S to Γ with $R_1 \circ H_1 \circ f(1) = x_0$ and the homotopy class $[R_1 \circ H_1 \circ f] = a_1 \cdots a_n$ in $\pi_1(\Gamma, x_0)$. Since k is contractible in N , $a_1 \cdots a_n = 1$ in $\pi_1(\Gamma, x_0)$. Since $k \cap v \neq \emptyset$, $n > 0$. Therefore there must be some cancellation and it follows that $n \geq 2$. Suppose $a_i a_{i+1} = 1$ for some $i = 1, \dots, n-1$. Let c_1 denote the arc component of $k \setminus P(v \times (-1, 1))$ which corresponds to this cancellation. Then c_1 is not a spanning arc in $(\text{Bd } N) \setminus P(v \times (-1, 1))$. The cyclic word $(a_{i+1}, \dots, a_n, a_1, \dots, a_i)$ is also associated with k . A similar argument to one above shows that $a_{i+1} \cdots a_n a_1 \cdots a_i = 1$ in $\pi_1(\Gamma, x_0)$. There is some cancellation in the word $a_{i+1} \cdots a_n a_1 \cdots a_i$ and a corresponding arc component c_2 of $k \setminus P(v \times (-1, 1))$ is not a spanning arc in $(\text{Bd } N) \setminus P(v \times (-1, 1))$ and is distinct from c_1 . Thus the Lemma is proved. \square

Section 3.7: Invariant homeomorphisms

Lemma 3.7.1: Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose $h: N \rightarrow N$ is a homeomorphism which is homotopic to the identity and $h(\text{Im } P) = \text{Im } P$. Then for each $i = 1, \dots, g$ and $\epsilon = \pm 1$, $h(P(v_i \times \{\epsilon\})) = P(v_i \times \{\epsilon\})$.

Proof of Lemma 3.7.1: Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose $h: N \rightarrow N$ is a homeomorphism which is homotopic to the identity and $h(\text{Im } P) = \text{Im } P$. Let $k = k_1 \cup \dots \cup k_g$ be a system of curves in $\text{Bd } N$ which is conjugate to v and is in general position with P . Let Γ be a core of N associated with P , let R be an associated retract, and let $\{[f_1], \dots, [f_g]\}$ be the basis for $\pi_1(\Gamma, x_0)$ defined in Section 3.2. Since there is a mod $\text{Im } P$ isotopy, $H: N \times I \rightarrow N$, such that $H_1 \circ h(x_0) = x_0$, it can be assumed without loss of generality that $h(x_0) = x_0$. Let S denote the unit circle in the complex numbers. Let k_i be one of the components of k . Choose a homeomorphism $f: S \rightarrow k_i$ of S onto k_i with $f(1) \notin \text{Im } P$, $f(\frac{2}{3}\pi i) \in P(v_i \times \{-1\})$, and $f(\frac{4}{3}\pi i) \in P(v_i \times \{1\})$. Then the cyclic word for k_i defined as in Section 3.2 with respect to the homeomorphism f is $([f_i])$. Let B denote the 3-cell $N \setminus P(v \times (-1, 1))$. Choose an embedding $g: [0, 1] \rightarrow B$ such that $g(0) = x_0$ and $(\text{Im } g) \cap \text{Bd } B = \{f(1)\}$. Define the map $\varphi: S \rightarrow N$ as follows. If $\exp(\frac{2}{3}\pi i t) \in S$ with $t \in [0, 3]$, let

$$\varphi(\exp \frac{2}{3}\pi i t) = \begin{cases} g(t) & \text{if } t \in [0, 1] , \\ f(\exp(2\pi i(t-1))) & \text{if } t \in [1, 2] , \\ g(3-t) & \text{if } t \in [2, 3] . \end{cases}$$

Then the homotopy class $[R_1 \circ \varphi] = [f_i]$ in $\pi_1(\Gamma, x_0)$.

Since h is homotopic to the identity, the homotopy class $[R_1 \circ h \circ \varphi] = [f_i]$ in $\pi_1(\Gamma, x_0)$. Since the image under h of a component of $\text{Im } P$ is a component of $\text{Im } P$,

$h(P(v_i \times [-1, 1])) = P(v_j \times [-1, 1])$ for some $j = 1, \dots, g$.

Since $k_i \cap P((\text{Bd } v_i) \times [-1, 1])$ is a spanning arc in the annulus $P((\text{Bd } v_i) \times [-1, 1])$ and $h(P((\text{Bd } v_i) \times [-1, 1])) = P((\text{Bd } v_j) \times [-1, 1])$, it follows that

$h(k_i) \cap P((\text{Bd } v_j) \times [-1, 1])$ is a spanning arc in the annulus $P((\text{Bd } v_j) \times [-1, 1])$. Since there is an isotopy $J: N \times I \rightarrow N$, which is invariant on $\{x_0\} \cup P(v \times \{-1, 1\})$ and such that $J_1 \circ h(k_i)$ is in general position with P , it can be assumed without loss of generality that $h(k_i)$ is in general position with P . Since $h \circ f(\exp \frac{4}{3} \pi i) \in P(v_j \times \{\epsilon\})$ for some $\epsilon = \pm 1$, it follows that the cyclic word for $h(k_i)$ defined as in Section 3.2 is $([f_j]^\epsilon)$ and the homotopy class $[R_1 \circ h \circ \varphi] = [f_j]^\epsilon$ in $\pi_1(\Gamma, x_0)$. Since $[R_1 \circ h \circ \varphi] = [f_i]$, it follows that $j = i$ and $\epsilon = 1$.

Therefore $h \circ f(\exp \frac{4}{3} \pi i) \in P(v_i \times \{1\})$ and $h \circ f(\exp \frac{2}{3} \pi i) \in P(v_i \times \{-1\})$. Since the images under h of the components of $P(v_1 \times \{1\}) \cup \dots \cup P(v_g \times \{1\}) \cup P(v_1 \times \{-1\}) \cup \dots \cup P(v_g \times \{-1\})$ are themselves components, it follows that $h(P(v_i \times \{1\})) = P(v_i \times \{1\})$ and $h(P(v_i \times \{-1\})) = P(v_i \times \{-1\})$. Thus the Lemma is proved. \square

Chapter 4: Special T-transformations

Section 4.1: The definition

Let N be a handlebody of genus $g > 0$, let v be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose k is a system of curves in $Bd N$ in general position with P . Denote the 2-cell with $2g - 1$ holes, $(Bd N) \setminus P(v \times (-1, 1))$, by $D(2g - 1)$. Suppose there is an arc component c of $k \setminus P(v \times (-1, 1))$ which is not a spanning arc in $D(2g - 1)$. Then $Bd c \subset P(Bd v_i \times \{\epsilon\})$ for some $i = 1, \dots, g$ and $\epsilon = \pm 1$. The arc c separates $D(2g - 1)$. Let $F(c)$ denote the closure of the complementary domain of c in $D(2g - 1)$ with the property that $F(c) \cap P(Bd v_i \times \{-\epsilon\}) = \emptyset$. Let $F'(c)$ denote the closure of the other complementary domain of c in $D(2g - 1)$. Then $F(c)$ is a 2-cell with n holes for some n , $0 \leq n \leq 2g - 2$, and $F'(c)$ is a 2-cell with $(2g - 1) - n$ holes. Let U be a regular neighborhood of $F(c) \cup P(Bd v_i \times \{\epsilon\})$ in $D(2g - 1)$ with respect to the subcomplex $k \setminus P(v \times (-1, 1))$. Let u denote the unique boundary component of U which lies in $Int D(2g - 1)$. Note that k is in general position with u in $Bd N$ and since $c \subset U \setminus u$, $\text{card}(u \cap k) < \text{card}(v_i \cap k)$. Choose a 2-cell v'_i properly embedded in the 3-cell $N \setminus P(v \times (-1, 1))$ with the property that $Bd v'_i = u$. Let $v' = (v \cup v'_i) \setminus v_i$. Note that $\text{card}(v' \cap k) = \text{card}(v \cap k) + \text{card}(u \cap k) - \text{card}(v_i \cap k)$.

Thus $\text{card}(v' \cap k) < \text{card}(v \cap k)$. Moreover, v' is the union of a collection of g mutually disjoint properly embedded 2-cells in N . Since

$[(\text{Bd } N) \setminus \text{Bd } v'_i] \cup P((\text{Bd } v_i) \times [-1, 1])$ is connected, it follows that $(\text{Bd } N) \setminus \text{Bd } v'$ is also connected. By a theorem in Section 1.8 v' is a system of meridian disks for N . The substitution $v \mapsto v'$ is called a special T-transformation with respect to k (along c) [W,3]. See also [Wa], [Z,1], [Z,2], [Z,3]. Note that k is in general position with $\text{Bd } v'$. The following Lemma is an immediate consequence of the above.

Lemma 4.1.1 [Z,1; p.236]: Let N be a handlebody of genus $g > 0$ and let v be a system of meridian disks for N . Suppose k is a system of curves in $\text{Bd } N$ in general position with $\text{Bd } v$. If k is not cyclically reduced with respect to v , then there is a special T-transformation $v \mapsto v'$ with respect to k such that $\text{card}(v' \cap k) < \text{card}(v \cap k)$.

Corollary 4.1.2: Let N be a handlebody of genus $g > 0$ and let v be a system of meridian disks for N . Suppose k is a system of curves in $\text{Bd } N$ in general position with $\text{Bd } v$. Then there is a finite sequence $v, v', \dots, v^{(n)}$ with $n \geq 0$ of systems of meridian disks for N such that $v^{(i)} \mapsto v^{(i+1)}$ is a special T-transformation with respect to k for each $i = 0, \dots, n-1$ and k is cyclically reduced with respect to $v^{(n)}$.

Proof of Corollary 4.1.2: Let N be a handlebody of genus $g > 0$ and let v be a system of meridian disks for N . Suppose k is a system of curves in BdN in general position with Bdv . Let $m = \text{card}(v \cap k)$. If $m = 0$, then k is cyclically reduced with respect to v and the Corollary is true. Let $p \geq 0$. As an induction hypothesis assume that if w is a system of meridian disks for N , k is in general position with Bdw , and $\text{card}(w \cap k) \leq p$, then there is a finite sequence $w, w', \dots, w^{(n)}$ with $n \geq 0$ of systems of meridian disks for N such that $w^{(i)} \mapsto w^{(i+1)}$ is a special T-transformation with respect to k for each $i = 0, \dots, n-1$ and k is cyclically reduced with respect to $w^{(n)}$. Suppose $m = p + 1$. If k is cyclically reduced with respect to v , then let $n = 0$ and v is itself the desired sequence of systems of meridian disks. If k is not cyclically reduced with respect to v , then by Lemma 4.1.1 there is a special T-transformation $v \mapsto v'$ of k and $\text{card}(v' \cap k) < \text{card}(v \cap k) = p + 1$. Therefore it follows from the induction hypothesis that there is a finite sequence $v, v', \dots, v^{(n)}$ with $n \geq 1$ of systems of meridian disks for N such that for each $i = 0, \dots, n-1$ $v^{(i)} \mapsto v^{(i+1)}$ is a special T-transformation with respect to k and k is cyclically reduced with respect to $v^{(n)}$. Thus the Corollary is proved. \square

Section 4.2: The proof of Lemma 2.5

Lemma 4.2.1: Let N be a handlebody of genus two, let w be a system of meridian disks for N , and let k be a system of curves in $Bd N$ which is conjugate to w . Suppose v is a system of meridian disks for N such that k is conjugate to v . Then there is an isotopy, $H: N \times I \rightarrow N$, which is invariant on k (and is not necessarily a mod k isotopy) such that $H_1(v) = w$.

Proof of Lemma 4.2.1: Let N be a handlebody of genus two, let $w = w_1 \cup w_{-1}$ be a system of meridian disks for N , and let $k = k_1 \cup k_{-1}$ be a system of curves in $Bd N$ which is conjugate to w . Suppose $v = v_1 \cup v_{-1}$ is a system of meridian disks for N and k is conjugate to v . Choose a 2-sided embedding of v in N , $P: v \times [-1, 1] \rightarrow N$, such that k is in general position with P . There is an isotopy, $J: N \times I \rightarrow N$, which is invariant on k such that $J_1(Bd w)$ is in general position with P . Define an isotopy, $J': N \times I \rightarrow N$, by $J'(x, t) = (J_t)^{-1}(x)$ for each $(x, t) \in N \times I$. It follows that J' is invariant on k . Define a 2-sided embedding of the system of meridian disks $J'_1(v)$ in N , $P': J'_1(v) \times [-1, 1] \rightarrow N$, by $P'(J'_1(x), t) = J'_1 \circ P(x, t)$ for each $x \in v$ and $t \in [-1, 1]$. It follows that $Bd w$ is in general position with P' . Therefore it can be assumed without loss of generality that $Bd w$ is in general position with P . Denote the 2-cell with 3 holes $(Bd N) \setminus P(v \times (-1, 1))$ by $D(3)$. Note that since each of k

and $Bd w$ is in general position with P and $k \cap w$ cannot contain an arc, it follows that $k \cap w \subset \text{Int } D(3)$. The proof of Lemma 2.4.1 is by induction on $\text{card}(v \cap Bd w)$. If $\text{card}(v \cap Bd w) = 0$, then $Bd w \subset \text{Int } D(3)$. Let F_1, F'_1 be the closures of the complementary domains of $Bd w_1$ in $D(3)$. Since $k_{-1} \cap Bd w_1 = \emptyset$, k_1 is in general position with $Bd w_1$ in $D(3)$, and $k_1 \cap Bd w_1$ is a single point, it follows that one of F_1, F'_1 is an annulus and the other is a 2-cell with 3 holes. Assume F_1 is the annulus. Since $k_{-1} \cap Bd w_{-1} \neq \emptyset$, it follows that $Bd w_{-1} \subset \text{Int } F'_1$. Since $k_1 \cap Bd w_{-1} = \emptyset$, k_{-1} is in general position with $Bd w_{-1}$ in $D(3)$, and $k_{-1} \cap Bd w_{-1}$ is a single point, it follows that one of the closures of the complementary domains of $Bd w_{-1}$ in F'_1 is an annulus, F_2 . It follows that there is an isotopy, $J': (Bd N) \times I \rightarrow Bd N$, which is invariant on k and such that $J'_1(Bd v) = Bd w$. By theorems in Section 1.3 and Section 1.8 it follows that there is an isotopy, $H: N \times I \rightarrow N$, which is invariant on k and such that $H_1(v) = w$. Thus the Lemma is true. Let $m \geq 0$. As an induction hypothesis assume that if v is a system of meridian disks for N , $P: v \times [-1, 1] \rightarrow N$ is a 2-sided embedding of v in N , k is conjugate to v , each of k and $Bd w$ is in general position with P , and $\text{card}(v \cap Bd w) \leq m$, then there is an isotopy, $H: N \times I \rightarrow N$, which is invariant on k and such that $H_1(v) = w$. Suppose v is a system of meridian disks for N , k is conjugate

to v , $P: v \times [-1,1] \rightarrow N$ is a 2-sided embedding of v in N , each of k and $Bd w$ is in general position with P , and $\text{card}(v \cap Bd w) = m+1$. Since $\text{card}(v \cap Bd w) \neq 0$, $v \cap Bd w_i \neq \emptyset$ for some $i = \pm 1$. By Lemma 3.6.1 there are two arc components c_1, c_2 of $(Bd w_i) \setminus P(v \times (-1,1))$ which are not spanning arcs. Since $k_i \cap Bd w_i$ is a single point, it follows that at least one of $k \cap c_1, k \cap c_2$ is empty. Assume $k \cap c_1 = \emptyset$. Since c_1 is not a spanning arc in $D(3)$, $Bd c_1 \subset P(Bd v_j \times \{\epsilon\})$ for some $j = \pm 1$ and $\epsilon = \pm 1$. It follows that $F(c_1)$ is either 2-cell or a 2-cell with 2 holes since $(k \setminus P(v \times (-1,1))) \cap c_1 = \emptyset$. In either case if U is a regular neighborhood of $P(Bd v_j \times \{\epsilon\}) \cup F(c_1)$ in $D(3)$ with respect to the subcomplex $(k \cup Bd w) \setminus P(v \times (-1,1))$ and u is the unique boundary component of U which lies in $\text{Int } D(3)$, then k is in general position with u in $Bd N$, $k \cap u$ consists of a single point, and this point lies in k_j . Therefore if v'_j is a properly embedded 2-cell in the 3-cell $N \setminus P(v \times (-1,1))$ with $Bd v'_j = u$ and $v \mapsto v' = (v \cup v'_j) \setminus v_j$ is the corresponding special T-transformation, then k is conjugate to v' . Furthermore the closure of one complementary domain of u in $D(3)$ is an annulus with boundary $u \cup P(Bd v_j \times \{-1\})$ or $u \cup P(Bd v_j \times \{1\})$. In either case it follows that there is an isotopy, $H: N \times I \rightarrow N$, such that H is invariant on k and $H_1(v) = v'$. Choose a 2-sided embedding, $P': v' \times [-1,1] \rightarrow N$, of v' in N

such that each of k and Bdw are in general position with P' . Since $\text{card}(v' \cap Bdw) < \text{card}(v \cap Bdw) = m+1$, by the inductive hypothesis there is an isotopy, $H' : N \times I \rightarrow N$, such that H' is invariant on k and $H'_1(v') = w$. Define the isotopy, $H'' : N \times I \rightarrow N$, as follows. If $(x, t) \in N \times I$,

$$\text{let } H''(x, t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ H'(H_1(x), 2t - 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

Then H'' is invariant on k and $H''_1(v) = w$. Thus the Lemma is proved. \square

Proof of Lemma 2.5: Lemma 2.5 is an immediate consequence of Lemma 4.2.1. \square

Chapter 5: Band changes

Section 5.1: Cyclically reduced systems of curves

Lemma 5.1.1 [Z, 2; p. 241]: Let N be a handlebody of genus $g \geq 2$ and let v be a system of meridian disks for N . Let v_i be one of the components of v . Suppose v'_i is a properly embedded 2-cell in N with $v \cap v'_i = \emptyset$ and $v' = (v \cup v'_i) \setminus v_i$ is a system of meridian disks for N . Suppose k is a system of curves in $Bd N$ which is in general position with $Bd(v \cup v'_i)$ and k is cyclically reduced with respect to v . Suppose for each component c of $k \setminus v$ that $\text{card}(c \cap v'_i) \leq 1$. Then k is cyclically reduced with respect to v' .

Proof of Lemma 5.1.1: Let N be a handlebody of genus $g \geq 2$ and let v be a system of meridian disks for N . Let v_i be one of the components of v . Suppose v'_i is a properly embedded 2-cell in N with $v \cap v'_i = \emptyset$ and $v' = (v \cup v'_i) \setminus v_i$ is a system of meridian disks for N . Suppose k is a system of curves in $Bd N$ which is in general position with $Bd(v \cup v'_i)$ and k is cyclically reduced with respect to v . Suppose for each component c of $k \setminus v$ that $\text{card}(c \cap v'_i) \leq 1$. Suppose k is not cyclically reduced with respect to v' . Then there is a 2-sided embedding of v' in N , $P': v' \times [-1, 1] \rightarrow N$, and an arc component d' of $k \setminus P'(v' \times (-1, 1))$ which is not a spanning arc in $(Bd N) \setminus P'(v' \times (-1, 1))$. Therefore

$Bd\,d = \{P'(x_1, \delta), P'(x_2, \delta)\}$ for some $\delta = \pm 1$ and x_1, x_2 distinct points of v' which lie in the same component of v' . There is an ϵ , $0 < \epsilon \leq 1$, such that $P'(v' \times [-\epsilon, \epsilon]) \cap v_i = \emptyset$. If $\delta = 1$, then $d' \cup P'(\{x_1, x_2\} \times [\epsilon, 1])$ is an arc component of $k \setminus P'(v' \times [-\epsilon, \epsilon])$ which is not a spanning arc in $(Bd\,N) \setminus P'(v' \times [-\epsilon, \epsilon])$. If $\delta = -1$, then $d' \cup P'(\{x_1, x_2\} \times [-1, -\epsilon])$ is an arc component of $k \setminus P'(v' \times [-\epsilon, \epsilon])$ which is not a spanning arc in $(Bd\,N) \setminus P'(v' \times [-\epsilon, \epsilon])$. If $Q': v' \times [-1, 1] \rightarrow N$ is defined by $Q'(x, t) = P'(x, t\epsilon)$ for each $(x, t) \in v' \times [-1, 1]$, then Q' is a 2-sided embedding of v' in N , k is in general position with k , and there is an arc component of $k \setminus Q'(v' \times (-1, 1))$ which is not a spanning arc in $(Bd\,N) \setminus Q'(v' \times (-1, 1))$. Therefore it can be assumed without loss of generality that $v_i \cap \text{Im } P' = \emptyset$. Choose a 2-sided embedding of $v \cup v'_i$ in N , $Q: (v \cup v'_i) \times [-1, 1] \rightarrow N$, such that $Q|(v' \times [-1, 1]) = P'$ and if $P: v \times [-1, 1] \rightarrow N$ is defined by $P = Q|(v \times [-1, 1])$, then k is in general position with P . Let the 2-cell with $2g-1$ holes, $(Bd\,N) \setminus P(v \times (-1, 1))$, be denoted by $D(2g-1)$ and let the 2-cell with $2g-1$ holes, $(Bd\,N) \setminus P'(v' \times (-1, 1))$, be denoted by $D'(2g-1)$. Let d' be an arc component of $k \setminus P'(v' \times (-1, 1))$ which is not a spanning arc in $D'(2g-1)$. Let $Bd\,d' = \{P'(x_1, \delta), P'(x_2, \delta)\}$ for some $\delta = \pm 1$ and distinct points x_1, x_2 which lie in the same component of v' . There are two cases to consider. Either

$Bd d' \subset P'(v_j \times \{\delta\})$ for some $j \neq i$ or $Bd d' \subset P'(v'_i \times \{\delta\})$.
 Suppose $Bd d' \subset P'(v_j \times \{\delta\})$ for some $j \neq i$. If $d' \cap v_i = \emptyset$,
 then d' is an arc component of $k \setminus P(v \times (-1, 1))$ which is
 not a spanning arc in $D(2g-1)$. This is impossible. If
 $\text{card}(d' \cap v_i) \geq 2$, then since $Bd v_i$ separates $D'(2g-1)$
 there is an arc component of $k \setminus P(v \times (-1, 1))$ contained in
 d' which is not a spanning arc in $D(2g-1)$. This is im-
 possible. If $\text{card}(d' \cap v_i) = 1$, then $Bd v_i$ separates
 $Bd d'$ in $D'(2g-1)$. This is impossible since $Bd d'$ is
 contained in a single boundary component of $D'(2g-1)$.
 Suppose on the other hand that $Bd d' \subset P'(v'_i \times \{\delta\})$. If
 $d' \cap v_i = \emptyset$, then $d' \subset D(2g-1)$. Let c denote the com-
 ponent of $k \setminus v$ which contains d' . Then $\text{card}(c \cap v'_i) \geq 2$,
 which is impossible. If $\text{card}(d' \cap v_i) \geq 2$, then since
 $Bd v_i$ separates $D'(2g-1)$ there is an arc component of
 $k \setminus P(v \times (-1, 1))$ contained in d' which is not a spanning
 arc in $D(2g-1)$. This is impossible. If $\text{card}(d' \cap v_i) = 1$,
 then $Bd v_i$ separates $Bd d'$ in $D'(2g-1)$. This is im-
 possible since $Bd d'$ is contained in a single boundary
 component of $D'(2g-1)$. Thus the Lemma is proved. \square

Section 5.2: Band changes

Let N be a handlebody of genus $g \geq 2$, let v be a
 system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$
 be a 2-sided embedding of v in N . Denote the 2-cell
 with $2g-1$ holes, $(Bd N) \setminus P(v \times (-1, 1))$ by $D(2g-1)$ and

the 3-cell, $N \setminus P(v \times (-1,1))$, by B . Let v_i, v_j be distinct components of v . Suppose c is a spanning arc in $D(2g-1)$ with $(Bd c) \cap P(v_i \times \{\epsilon\}) \neq \emptyset$ and $(Bd c) \cap P(v_j \times \{\delta\}) \neq \emptyset$ for some $\epsilon = \pm 1, \delta = \pm 1$.

Choose a regular neighborhood U of $c \cup P((Bd v_i) \times \{\epsilon\}) \cup P((Bd v_j) \times \{\delta\})$ in $D(2g-1)$. Let u denote the unique boundary component of U which lies in $\text{Int } D(2g-1)$. Choose a properly embedded 2-cell $v'_i = u$. Let $v' = (v \cup v'_i) \setminus v_i$. Since $(D(2g-1) \setminus Bd v'_i) \cup P((Bd v_i) \times [-1,1])$ is connected, it follows that $(Bd N) \setminus Bd v'$ is also connected. Therefore by a theorem in Section 1.8 v' is a system of meridian disks for N . The substitution $v \mapsto v'$ is called a band change (along c) [S]. See also [W,1], [B].

Section 5.3: Genus two special T-transformations

Lemma 5.3.1: Let N be a handlebody of genus two, let v be a system of meridian disks for N , let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N , and let k be a system of curves in $Bd N$ in general position with P . Suppose $v \mapsto v'$ is a special T-transformation with respect to k . Then $v' \mapsto v$ is a band change or there is an isotopy of N , $H: N \times I \rightarrow N$, such that $H_1(v') = v$.

Proof of Lemma 5.3.1: Let N be a handlebody of genus two, let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N ,

let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N , and let k be a system of curves in $Bd N$ in general position with P . Suppose $v \mapsto v'$ is a special T-transformation with respect to k along an arc component c of $k \setminus P(v \times (-1, 1))$ with $Bd c \subset P(v_i \times \{\epsilon\})$ for some $i = \pm 1$ and $\epsilon = \pm 1$. Denote the 2-cell with 3 holes, $(Bd N) \setminus P(v \times (-1, 1))$, by $D(3)$. It follows that $F(c)$ is a 2-cell with 0, 1, or 2 holes. Suppose $F(c)$ is a 2-cell with 0-holes. Let U be the regular neighborhood of $F(c) \cup P((Bd v_i) \times \{\epsilon\})$ whose unique boundary component which lies in $Int D(3)$ is $Bd v'_i$. If $\epsilon = 1$, then $U \cup P((Bd v_i) \times [0, 1])$ is an annulus in $Bd N$. If $\epsilon = -1$, then $U \cup P((Bd v_i) \times [-1, 0])$ is an annulus in $Bd N$. In either case it follows that there is an isotopy, $J: (Bd N) \times I \rightarrow Bd N$, such that $J_1(Bd v'_i) = Bd v$. By theorems in Section 1.3 and Section 1.8 it follows that there is an isotopy, $H: N \times I \rightarrow N$, such that $H_1(v') = v$. Suppose $F(c)$ is a 2-cell with 2 holes. If $\epsilon = 1$, then $F'(c) \cup P((Bd v_i) \times [0, 1])$ is an annulus. If $\epsilon = -1$, then $F'(c) \cup P((Bd v_i) \times [-1, 0])$ is an annulus. As above it follows that there is an isotopy, $H: N \times I \rightarrow N$, such that $H_1(v') = v$. Suppose finally that $F(c)$ is a 2-cell with 1 hole. Choose a 2-sided embedding of $v \cup v'_i$ in N , $Q: (v \cup v'_i) \times [-1, 1] \rightarrow N$, such that $P = Q|_{(v \times [-1, 1])}$. Since $F(c)$ is a 2-cell with 1 hole, it follows that U is a 2-cell with 2 holes. It then follows that

$(\text{Bd } N) \setminus Q((v \cup v'_i) \times (-1, 1))$ is the disjoint of two 2-cells with 2 holes each. Define a 2-sided embedding of v' in N , $P': v' \times [-1, 1] \rightarrow N$, by $P' = Q|_{(v' \times [-1, 1])}$. Denote the 2-cell with 3 holes, $(\text{Bd } N) \setminus P'(v' \times (-1, 1))$, by $D'(3)$. Then $\text{Bd } v_i \subset \text{Int } D'(3)$. Since $(\text{Bd } N) \setminus Q((v \cup v'_i) \times (-1, 1))$ is the disjoint union of two 2-cells with 2 holes, it follows that the closures of the complementary domains of $\text{Bd } v_i$ in $D'(3)$ are 2-cells with 2 holes. Let U' denote the closure of one of the complementary domains of $\text{Bd } v_i$ in $D'(3)$. Since $\text{Bd } v_i$ does not separate $\text{Bd } N$, the two boundary components of U' which are different from $\text{Bd } v_i$ must correspond to different components of v' . Hence $\text{Bd } U'$ consists of $\text{Bd } v_i$, $P((\text{Bd } v'_i) \times \{\epsilon_1\})$, and $P((\text{Bd } v_{-i}) \times \{\epsilon_2\})$ for some $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 2$. Let d be a spanning arc in U' with $\text{Bd } d \subset P(v'_i \times \{\epsilon_1\}) \cup P(v_{-i} \times \{\epsilon_2\})$. Then U' is a regular neighborhood of $d \cup P((\text{Bd } v'_i) \times \{\epsilon_1\}) \cup P((\text{Bd } v_{-i}) \times \{\epsilon_2\})$ in $D'(3)$. It follows that $v' \mapsto v$ is a band change along d . Thus the Lemma is proved. \square

Section 5.4: Genus two band changes

Lemma 5.4.1 [Z, 2; p. 239]: Let N be a handlebody of genus two, let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Denote the 2-cell with 3 holes, $(\text{Bd } N) \setminus P(v \times (-1, 1))$, by $D(3)$. Suppose k is a system

of curves in $Bd N$ which is cyclically reduced with respect to P and no component of k is contractible in N . Then for each $\epsilon = \pm 1$ and $\delta = \pm 1$ $k \setminus P(v \times (-1, 1))$ does not separate $P(Bd v_{-1}) \times \{\epsilon\}$ from $P((Bd v_1) \times \{\delta\})$ in $D(3)$.

Proof of Lemma 5.4.1: Let N be a handlebody of genus two, let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N , and let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Denote the 2-cell with 3 holes, $(Bd N) \setminus P(v \times (-1, 1))$, by $D(3)$. Suppose k is a system of curves in $Bd N$ which is cyclically reduced with respect to P and no component of k is contractible in N . Suppose for some $\epsilon = \pm 1$ and $\delta = \pm 1$ that $k \setminus P(v \times (-1, 1))$ separates $P((Bd v_{-1}) \times \{\epsilon\})$ from $P((Bd v_1) \times \{\delta\})$ in $D(3)$. Then there is a minimal collection c_1, \dots, c_n with $n \geq 1$ of components of $k \setminus P(v \times (-1, 1))$ whose union separates $P((Bd v_{-1}) \times \{\epsilon\})$ from $P((Bd v_1) \times \{\delta\})$ in $D(3)$. Each c_i is an arc since no component of k is contractible in N . Each c_i is a spanning arc in $D(3)$ since k is cyclically reduced with respect to P . Suppose $c_n \cap (P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\})) \neq \emptyset$. Since the union of the collection c_1, \dots, c_{n-1} does not separate $P((Bd v_{-1}) \times \{\epsilon\})$ from $P((Bd v_1) \times \{\delta\})$ in $D(3)$, there is a properly embedded arc d in $D(3)$ in general position with c_n such that $d \cap (c_1 \cup \dots \cup c_{n-1}) = \emptyset$ and d is a spanning arc in $D(3)$ with $Bd d \subset P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\})$. Let U be a regular neighborhood of c_n in $D(3)$ with respect to the

subcomplex $c_1 \cup \dots \cup c_n \cup d$. Then $d \cup Cl((\text{Int } D(3)) \cap Bd U)$ contains an arc d' properly embedded in $D(3)$ and d' is a spanning arc in $D(3)$ with $Bd d' \subset P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\})$. Since $d' \cap (c_1 \cup \dots \cup c_n) = \emptyset$, the union $c_1 \cup \dots \cup c_n$ does not separate $P((Bd v_{-1}) \times \{\epsilon\})$ from $P((Bd v_1) \times \{\delta\})$ in $D(3)$. This is a contradiction. It follows that $Bd c_n \subset P(v_{-1} \times \{-\epsilon\}) \cup P(v_1 \times \{-\delta\})$.

Define a function $f: (k \setminus P(v \times (-1, 1))) \cap (P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\})) \rightarrow (k \setminus P(v \times (-1, 1))) \cap (P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{-\delta\}))$ as follows. If $x \in (k \setminus P(v \times (-1, 1))) \cap (P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\}))$, let c be the component of $k \setminus P(v \times (-1, 1))$ with $x \in Bd c$. Define $f(x)$ by $Bd c = \{x, f(x)\}$. Since no component of $k \setminus P(v \times (-1, 1))$ is a spanning arc in $D(3)$ with its boundary in $P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\})$, it follows that $f(x) \in P(v_{-1} \times \{-\epsilon\}) \cup P(v_1 \times \{-\delta\})$. Therefore f is well-defined. It also follows that f is an injection. Since $(\text{Im } f) \cap Bd c_n = \emptyset$, $\text{card}((k \setminus P(v \times (-1, 1))) \cap (P(v_{-1} \times \{\epsilon\}) \cup P(v_1 \times \{\delta\}))) < \text{card}((k \setminus P(v \times (-1, 1))) \cap (P(v_{-1} \times \{-\epsilon\}) \cup P(v_1 \times \{-\delta\})))$. Since these cardinalities are equal, this is a contradiction. Thus the Lemma is proved. \square

Lemma 5.4.2 [Z, 2; p. 240]: Let N be a handlebody of genus two, let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N , let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N , and let k be a system of curves in $Bd N$ which is cyclically reduced with respect to P and such that no component

of k is contractible in N . Let $\epsilon = \pm 1$, $\delta = \pm 1$, and $i = \pm 1$. Then there is a spanning arc d in $(Bd N) \setminus P(v \times (-1, 1))$ with $Bd d \subset P(v_1 \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$ and a 2-cell v'_i properly embedded in the 3-cell $N \setminus P(v \times (-1, 1))$ such that $v \mapsto v' = (v \cup v'_i) \setminus v_i$ is a band change along d and k is cyclically reduced with respect to v' .

Proof of Lemma 5.4.2: Let N be a handlebody of genus two, let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N , let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N , and let k be a system of curves in $Bd N$ which is cyclically reduced with respect to P and such that no component of k is contractible in N . Fix $\epsilon = \pm 1$, $\delta = \pm 1$, and $i = \pm 1$. Since no component of $k \setminus P(v \times (-1, 1))$ is contractible in N , each component of $k \setminus P(v \times (-1, 1))$ is an arc. Let m denote the cardinality of the set $\{c \mid c \text{ is a component of } k \setminus P(v \times (-1, 1)) \text{ and } Bd c \subset P(v_i \times \{\epsilon\}) \cup P(v_{-i} \times \{\delta\})\}$. Denote the 2-cell with 3 holes, $(Bd N) \setminus P(v \times (-1, 1))$, by $D(3)$. Suppose $m = 0$. By Lemma 5.4.1 $k \setminus P(v \times (-1, 1))$ does not separate $P((Bd v_i) \times \{\epsilon\})$ from $P((Bd v_{-i}) \times \{\delta\})$ in $D(3)$. Then there is a properly embedded arc d in $D(3)$ which is a spanning arc, $d \cap (k \setminus P(v \times (-1, 1))) = \emptyset$, and $Bd d \subset P(v_i \times \{\epsilon\}) \cup P(v_{-i} \times \{\delta\})$. Let U be a regular neighborhood of $d \cup P((Bd v_i) \times \{\epsilon\}) \cup P((Bd v_{-i}) \times \{\delta\})$ in $D(3)$ with respect to the subcomplex $d \cup (k \setminus P(v \times (-1, 1)))$. Let u be the unique boundary

component of U which lies in $\text{Int } D(3)$. Then k is in general position with $u \cup \text{Bd } v$. Also if c is a component of $k \setminus v$, $\text{card}(c \cap u) = 1$. Let v'_i be a properly embedded 2-cell in $N \setminus P(v \times (-1, 1))$ with $\text{Bd } v'_i = u$. Then $v' = (v \cup v'_i) \setminus v_i$ is a system of meridian disks for N , $v \mapsto v'$ is a band change along d , and by Lemma 5.1.1 k is cyclically reduced with respect to v' . Suppose $m = 1$. Then there is a component d of $k \setminus P(v \times (-1, 1))$ with $\text{Bd } d \subset P(v_i \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$. Let U be a regular neighborhood of $d \cup P(v_i \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$ in $D(3)$ with respect to the subcomplex $k \setminus P(v \times (-1, 1))$. Let u be the unique boundary component of U which lies in $\text{Int } D(3)$. Then k is in general position with $u \cup \text{Bd } v$. Also if c is a component of $k \setminus v$, then $\text{card}(c \cap u) \leq 1$. Therefore as above there is a band change $v \mapsto v'$ along d with k cyclically reduced with respect to v' . Suppose $m > 1$. Then there exists a component d of $k \setminus P(v \times (-1, 1))$ with $\text{Bd } d \subset P(v_i \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$. Suppose d' is a component of $k \setminus P(v \times (-1, 1))$ with $\text{Bd } d' \subset P(v_i \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$ and d' is distinct from d . Then the closure of one complementary domain of $d \cup d'$ in $D(3)$ is a 2-cell, for otherwise $k \setminus P(v \times (-1, 1))$ would separate $P(v_i \times \{-\epsilon\})$ from $P(v_{-1} \times \{-\delta\})$ in $D(3)$ which would violate Lemma 5.4.1. Consequently there exist d_1, d_2 components of $k \setminus P(v \times (-1, 1))$ with $\text{Bd } d_j \subset P(v_i \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$ for each $j = 1, 2$ and the closure, F , of a complementary

domain of $d_1 \cup d_2$ in $D(3)$ is a 2-cell with $d_1 \cup d_2 \subset \text{Bd } F$, and F contains each component d' of $k \setminus P(v \times (-1, 1))$ with $\text{Bd } d' \subset P(v_i \times \{\epsilon\}) \cup P(v_{-i} \times \{\delta\})$. Let U be a regular neighborhood of $F \cup P(v_i \times \{\epsilon\}) \cup P(v_{-i} \times \{\delta\})$ in $D(3)$ with respect to the subcomplex $k \setminus P(v \times (-1, 1))$. Let u be the unique boundary component of U which lies in $\text{Int } D(3)$. Then k is in general position with $u \cup \text{Bd } v$. Also if c is a component of $k \setminus v$, then $\text{card}(c \cap u) \leq 1$. Since U is a regular neighborhood of $d \cup P(v_i \times \{\epsilon\}) \cup P(v_{-i} \times \{\delta\})$, as above there is a band change $v \mapsto v'$ along d with k cyclically reduced with respect to v' . Thus the Lemma is proved. \square

Chapter 6: Two-cells with n holes

Section 6.1: One-subcomplexes

Lemma 6.1.1: Let $D(n)$ be a 2-cell with n holes with boundary components B, B_1, \dots, B_n . Let U be a nonempty subset of B which is open in B . Then there is a collection $\alpha_1, \dots, \alpha_n$ of mutually disjoint spanning arcs in $D(n)$ with $Bd\alpha_i \subset U \cup B_i$ for each $i = 1, \dots, n$.

Proof of Lemma 6.1.1: Let $D(n)$ be a 2-cell with n holes with boundary components B, B_1, \dots, B_n . Let U be a nonempty subset of B which is open in B . If $n = 0$, then there is nothing to prove. If $n = 1$, let α_1 be any spanning arc in $D(n)$ with $Bd\alpha_1 \subset U \cup B_1$. Thus the Lemma is true. Let $k \geq 1$. As an induction hypothesis assume that if $D(k)$ is a 2-cell with k holes with boundary components B, B_1, \dots, B_k and U is a nonempty subset of B which is open in B , then there is a collection $\alpha_1, \dots, \alpha_k$ of mutually disjoint spanning arcs in $D(k)$ with $Bd\alpha_i \subset U \cup B_i$ for each $i = 1, \dots, k$. Suppose $n = k + 1$. Since $D(n)$ is arcwise connected, there is a spanning arc, α_n , in $D(n)$ with $Bd\alpha_n \subset U \cup B_n$. Let $P: \alpha_n \times [-1, 1] \rightarrow D(n)$ be a 2-sided embedding of α_n in $D(n)$ with $P((Bd\alpha_n) \times [-1, 1]) \subset U$. Then $D(n) \setminus P(\alpha_n \times (-1, 1))$ is a 2-cell with k holes. Denote $D(n) \setminus P(\alpha_n \times (-1, 1))$ by F . The boundary components of F are B_1, \dots, B_{n-1} , and B' ,

where $B' = (B \cup B_n \cup \text{Im } P) \setminus P(\alpha_n \times (-1, 1))$. Let $U' = U \setminus \text{Im } P$. Then U' is a nonempty subset of B' which is open in B' . By the induction hypothesis there is a collection $\alpha_1, \dots, \alpha_{n-1}$ of mutually disjoint spanning arcs in F with $\text{Bd } \alpha_i \subset U' \cup B_i$ for each $i = 1, \dots, n-1$. Then $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$ is a collection of mutually disjoint spanning arcs in B with $\text{Bd } \alpha_i \subset U \cup B_i$ for each $i = 1, \dots, n$. Thus the Lemma is proved. \square

Corollary 6.1.2: Suppose $D(n)$ is a 2-cell with n holes with boundary components B, B_1, \dots, B_n . If α is a spanning arc in $D(n)$ with $\text{Bd } \alpha \subset B \cup B_n$, then there is a collection $\alpha_1, \dots, \alpha_n$ of mutually disjoint spanning arcs with $\alpha_n = \alpha$ and $\text{Bd } \alpha_i \subset B \cup B_i$ for each $i = 1, \dots, n$.

Proof of Corollary 6.1.2: Let $D(n)$ be a 2-cell with n holes with boundary components B, B_1, \dots, B_n . Suppose α is a spanning arc in $D(n)$ with $\text{Bd } \alpha \subset B \cup B_n$. Then $n \geq 1$. Let $P: \alpha \times [-1, 1] \rightarrow D(n)$ be a 2-sided embedding of α in $D(n)$. Then $D(n) \setminus P(\alpha \times (-1, 1))$ is a 2-cell with $n-1$ holes with boundary components B_1, \dots, B_{n-1} , and B' where $B' = (B \cup B_n \cup \text{Im } P) \setminus P(\alpha \times (-1, 1))$. Let $U' = B \setminus \text{Im } P$. Then U' is a nonempty subset of B' which is open in B' . By Lemma 6.1.1 there is a collection $\alpha_1, \dots, \alpha_{n-1}$ of mutually disjoint spanning arcs in $D(n) \setminus P(\alpha \times (-1, 1))$ with $\text{Bd } \alpha_i \subset U' \cup B_i$ for each $i = 1, \dots, n-1$. Let $\alpha_n = \alpha$. Then $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$ is the desired collection. Thus the Lemma is proved. \square

Section 6.2: Twists of $D(n)$

If $D(n)$ is a 2-cell with n holes, A is an annulus embedded in $D(n)$, and $h: D(n) \rightarrow D(n)$ is a homeomorphism such that for each $x \in D(n) \setminus \text{Int } A$ $h(x) = x$, then h is called a simple twist of $D(n)$.

If $D(n)$ is a 2-cell with n holes and for each $i = 1, \dots, k$ $h_i: D(n) \rightarrow D(n)$ is a simple twist of $D(n)$ or is a homeomorphism of $D(n)$ which is mod $\text{Bd } D(n)$ isotopic to the identity, then the composition, $h_k \circ \dots \circ h_1$, is called a twist of $D(n)$.

Lemma 6.2.1: Let S denote the unit circle in the complex numbers. Suppose α is a spanning arc in the annulus, $S \times [0, 1]$, and $\text{Bd } \alpha = \{(1, 0), (1, 1)\}$. Then there is a homeomorphism, $h: S \times [0, 1] \rightarrow S \times [0, 1]$, such that $h|_{\text{Bd}(S \times [0, 1])}$ is the identity and $h(\alpha) = \{1\} \times [0, 1]$.

Proof of Lemma 6.2.1: Suppose α is a spanning arc in the annulus, $S \times [0, 1]$, and $\text{Bd } \alpha = \{(1, 0), (1, 1)\}$. The map, $E: \mathbb{R} \times [0, 1] \rightarrow S \times [0, 1]$, defined by $E(t, u) = (\exp 2\pi i t, u)$ for each $(t, u) \in \mathbb{R} \times [0, 1]$ is a projection map for the universal covering space, $\mathbb{R} \times [0, 1]$, of $S \times [0, 1]$ onto $S \times [0, 1]$. Let c_0 denote the component of $E^{-1}(\alpha)$ with $(0, 0) \in \text{Bd } c_0$. It follows that $\text{Bd } c_0 = \{(0, 0), (n, 1)\}$ for some integer n . Let $c_1 = \{(t, u) \in \mathbb{R} \times [0, 1] \mid (t-1, u) \in c_0\}$. Since c_1 is a spanning

arc in $R \times [0,1]$ and $E(c_1) = \alpha$, it follows that c_1 is also a component of $E^{-1}(\alpha)$. The closure of one complementary domain of $c_0 \cup c_1$ in $R \times [0,1]$ is a 2-cell. Denote this 2-cell by D . Then $\text{Bd } D = c_0 \cup c_1 \cup ([0,1] \times \{0\}) \cup ([n, n+1] \times \{1\})$. Choose a homeomorphism, $F: D \rightarrow [0,1] \times [0,1]$, such that $F(c_0) = \{0\} \times [0,1]$, $F(c_1) = \{1\} \times [0,1]$, $F|([0,1] \times \{0\})$ is the identity, and $F|([n, n+1] \times \{1\})$ is defined by $F(t, 1) = (t-n, 1)$ for each $t \in [n, n+1]$. Define a homeomorphism, $h: S \times [0,1] \rightarrow S \times [0,1]$, as follows. If $x \in S \times [0,1]$, let y be the unique element of $D \setminus c_1$ for which $E(y) = x$. Define $h(x) = E \circ F(y)$. Then h is the desired homeomorphism. Thus the Lemma is proved. \square

Lemma 6.2.2: Let A be an annulus and suppose α_1, α_2 are spanning arcs in A with $\text{Bd } \alpha_1 = \text{Bd } \alpha_2$. Then there is a homeomorphism, $h: A \rightarrow A$, such that $h|(\text{Bd } A)$ is the identity and $h(\alpha_1) = \alpha_2$.

Proof of Lemma 6.2.2: Let A be an annulus and suppose α_1, α_2 are spanning arcs in A with $\text{Bd } \alpha_1 = \text{Bd } \alpha_2$. Let $f: A \rightarrow S \times [0,1]$ be a homeomorphism. Then there is a homeomorphism, $g: S \times [0,1] \rightarrow S \times [0,1]$, which is isotopic to the identity such that $g \circ f(\text{Bd } \alpha_1) = \{(1,0), (1,1)\}$. Therefore by Lemma 6.2.1 there are homeomorphisms, $h_i: S \times [0,1] \rightarrow S \times [0,1]$, for each $i=1,2$ such that $h_i \circ g \circ f(\alpha_i) = \{1\} \times [0,1]$ and $h_i|(\text{Bd } S \times [0,1])$ is the

identity. Let $h: A \rightarrow A$ be the homeomorphism $h = f^{-1} \circ g^{-1} \circ (h_2)^{-1} \circ h_1 \circ g \circ f$. Then h is the desired homeomorphism. Thus the Lemma is proved. \square

Lemma 6.2.3: Let $D(n)$ be a 2-cell with $n \geq 2$ holes with boundary components B, B_1, B_2, \dots, B_n . Suppose β_2, \dots, β_n is a collection of mutually disjoint spanning arcs in $D(n)$ with $Bd\beta_i \subset B \cup B_i$ for each $i = 2, \dots, n$. Suppose α_1 is a spanning arc in $D(n)$ with $Bd\alpha_1 \subset B \cup B_1$ and suppose that α_1 is in general position with $\beta_2 \cup \dots \cup \beta_n$. Then there is a twist of $D(n)$, $T: D(n) \rightarrow D(n)$, such that $T(\alpha_1) \cap (\beta_2 \cup \dots \cup \beta_n) = \emptyset$.

Proof of Lemma 6.2.3: Let $D(n)$ be a 2-cell with $n \geq 2$ holes with boundary components B, B_1, B_2, \dots, B_n . Suppose β_2, \dots, β_n is a collection of mutually disjoint spanning arcs in $D(n)$ with $Bd\beta_i \subset B \cup B_i$ for each $i = 2, \dots, n$. Suppose α_1 is a spanning arc in $D(n)$ with $Bd\alpha_1 \subset B \cup B_1$ and suppose that α_1 is in general position with $\beta_2 \cup \dots \cup \beta_n$. Let m denote $\text{card}(\alpha_1 \cap (\beta_2 \cup \dots \cup \beta_n))$. If $m = 0$, let $T = \text{identity}$. Thus the Lemma is true. Let $k \geq 1$. As an induction hypothesis assume the Lemma is true if $m < k$. Suppose $m = k$. Choose an embedding, $P: [0, k+1] \times [-1, 1] \rightarrow D(n)$, such that $P([0, k+1] \times \{0\}) = \alpha_1$, $P(0, 0) \in B_1$, $(\text{Im } P) \cap Bd D(n) = P(\{0, k+1\} \times [-1, 1])$ and $(\text{Im } P) \cap (\beta_2 \cup \dots \cup \beta_n) = P(\{1, 2, \dots, k\} \times [-1, 1])$. Since $\alpha_1 \cap (\beta_2 \cup \dots \cup \beta_n) \neq \emptyset$, There exists a component $\bar{\beta}$ of

$(\beta_2 \cup \cdots \cup \beta_n) \setminus P([0, k+1] \times (-1, 1))$ which has one endpoint in $B_2 \cup \cdots \cup B_n$ and the other endpoint equal to $P(j, \epsilon)$ for some integer j , $0 < j < k+1$, and $\epsilon = \pm 1$. It can be assumed without loss of generality, by perhaps reparameterizing P , that $\epsilon = 1$. There exists a pair of disjoint arcs C_1, C_2 properly embedded in

$D(n) \setminus P([0, k+1] \times (-1, 1))$ with $\text{Bd } C_1 = \{P(\frac{1}{3}, -1), P(\frac{1}{3}, 1)\}$, $\text{Bd } C_2 = \{P(j - \frac{1}{3}, 1), P(j + \frac{1}{3}, 1)\}$, and

$(C_1 \cup C_2) \cap (\beta_2 \cup \cdots \cup \beta_n) = \emptyset$. Define an arc $\alpha'_1 = P([0, \frac{1}{3}] \times \{0\}) \cup P(\{\frac{1}{3}\} \times [-1, 0]) \cup C_1 \cup P([\frac{1}{3}, j - \frac{1}{3}] \times \{1\}) \cup C_2 \cup P([j + \frac{1}{3}, k+1] \times \{0\})$. Then $\alpha'_1 \setminus P([0, k+1] \times (-1, \frac{1}{2}))$ is a properly embedded arc in the 2-cell with $n-1$ holes $D(n) \setminus P([0, k+1] \times (-1, \frac{1}{2}))$. Let U be a regular neighborhood of $\alpha'_1 \setminus P([0, k+1] \times (-1, \frac{1}{2}))$ in $D(n) \setminus P([0, k+1] \times (-1, \frac{1}{2}))$ with respect to the subcomplex $(\beta_2 \cup \cdots \cup \beta_n) \setminus P([0, k+1] \times (-1, \frac{1}{2}))$. Then U is a 2-cell. Therefore $U \cup P([0, k+1] \times [-1, \frac{1}{2}])$ is an annulus which is embedded in $D(n)$ and α_1, α'_1 are spanning arcs in this annulus with $\text{Bd } \alpha_1 = \text{Bd } \alpha'_1$. By Lemma 6.2.2 there is a twist of $D(n)$, $T: D(n) \rightarrow D(n)$, such that $T(\alpha_1) = \alpha'_1$. Now $\alpha'_1 \cap (\beta_2 \cup \cdots \cup \beta_n)$ contains $k-1$ points and α'_1 is in general position with $\beta_2 \cup \cdots \cup \beta_n$. By the induction hypothesis there is a twist of $D(n)$, $T': D(n) \rightarrow D(n)$, such that $T'(\alpha'_1) \cap (\beta_2 \cup \cdots \cup \beta_n) = \emptyset$. Then $T' \circ T$ is the desired twist of $D(n)$. Thus the lemma is proved. \square

Lemma 6.2.4: Let $D(n)$ be a 2-cell with n holes and suppose α, β are spanning arcs in $D(n)$ with $Bd \alpha = Bd \beta$. Then there is a twist $T: D(n) \rightarrow D(n)$ such that $T(\alpha) = \beta$.

Proof of Lemma 6.2.4: Let $D(n)$ be a 2-cell with n holes and suppose α, β are spanning arcs in $D(n)$ with $Bd \alpha = Bd \beta$. If $n=0$, then $D(n)$ is a 2-cell, $D(n)$ contains no spanning arcs, and there is nothing to prove. If $n=1$, then $D(n)$ is an annulus and the desired twist of $D(n)$ is given by Lemma 6.2.2. Let $k \geq 1$. As an induction hypothesis assume the Lemma is true for $n=k$. Suppose $n=k+1$. Let B, B_1, \dots, B_n be the boundary components of $D(n)$ such that $Bd \beta \subset B \cup B_1$. By Corollary 6.2.1 there is a collection β_1, \dots, β_n of mutually disjoint spanning arcs in $D(n)$ with $\beta_1 = \beta$ and $Bd \beta_i \subset B \cup B_i$ for each $i=1, \dots, n$. Since there is a homeomorphism, $h: D(n) \rightarrow D(n)$, which is mod $Bd D(n)$ isotopic to the identity such that $h(\alpha)$ is in general position with $\beta_2 \cup \dots \cup \beta_n$, it is without loss of generality that α can be assumed in general position with $\beta_2 \cup \dots \cup \beta_n$. By Lemma 6.2.3 there is a twist of $D(n)$, $T: D(n) \rightarrow D(n)$, such that $T(\alpha) \cap (\beta_2 \cup \dots \cup \beta_n) = \emptyset$. Let U be a regular neighborhood of $\beta_2 \cup \dots \cup \beta_n$ in $D(n)$ with respect to the subcomplex $T(\alpha) \cup \beta = T(\alpha) \cup \beta_1$. Since $Cl(D(n) \setminus U)$ is an annulus and $T(\alpha), \beta$ are spanning arcs in this annulus with $Bd T(\alpha) = Bd \beta$, by Lemma 6.2.2 there

is a twist of $D(n)$, $T': D(n) \rightarrow D(n)$, such that
 $T' \circ T(\alpha) = \beta$. Thus the Lemma is proved. \square

Lemma 6.2.5: Let $D(n)$ be a 2-cell with n holes and suppose $h: D(n) \rightarrow D(n)$ is a homeomorphism such that $h|_{Bd D(n)}$ is the identity. Then h is a twist of $D(n)$.

Proof of Lemma 6.2.5: Let $D(n)$ be a 2-cell with n holes and suppose $h: D(n) \rightarrow D(n)$ is a homeomorphism such that $h|_{Bd D(n)}$ is the identity. If $n=0$, then $D(n)$ is a 2-cell. The conclusion of the Lemma is provided by a theorem in Section 1.3. Let $k \geq 0$. As an induction hypothesis assume the Lemma is true for $n=k$. Suppose $n=k+1$.

Let B, B_1, \dots, B_n be the boundary components of $D(n)$. Since $D(n)$ is arcwise connected, there is a spanning arc α in $D(n)$ with $Bd \alpha \subset B \cup B_1$. Then $h(\alpha)$ is also a spanning arc in $D(n)$ and $Bd h(\alpha) = Bd \alpha$. By Lemma 6.2.4 there is a twist of $D(n)$, $T: D(n) \rightarrow D(n)$, such that $T \circ h(\alpha) = \alpha$. Let $P: \alpha \times [-1, 1] \rightarrow D(n)$ be a 2-sided embedding of α in $D(n)$. There is a mod $Bd D(n)$ isotopy, $H: D(n) \times I \rightarrow D(n)$, such that $H_1 \circ T \circ h(Im P) = Im P$. Moreover, there is a mod $Bd D(n)$ isotopy, $H': D(n) \times I \rightarrow D(n)$, which is invariant on $P(\alpha \times \{-1, 1\})$ such that $H'_1 \circ H_1 \circ T \circ h|_{Bd(Im P)}$ is the identity. By a theorem in Section 1.3 the homeomorphism $H'_1 \circ H_1 \circ T \circ h|_{Im P}: Im P \rightarrow Im P$ is mod $Bd(Im P)$ isotopic to the identity. Since $D(n) \setminus P(\alpha \times (-1, 1))$ is a 2-cell with $n-1=k$ holes and

$H_1' \circ H_1 \circ T \circ h|_{Bd(D(n) \setminus P(\alpha \times (-1,1)))}$ is the identity, by the inductive hypothesis $H_1' \circ H_1 \circ T \circ h|(D(n) \setminus P(\alpha \times (-1,1)))$ is a twist of $D(n) \setminus P(\alpha \times (-1,1))$. It follows that h is a twist of $D(n)$. Thus the Lemma is proved. \square

Chapter 7: The proof of Lemma 2.6

Proof of Lemma 2.6: Let N be a handlebody of genus g and let v be a system of meridian disks for N . Suppose $h: N \rightarrow N$ is an orientation-preserving homeomorphism which is homotopic to the identity and $h(v) = v$. If $g = 0$, then a theorem in Section 1.4 states that an orientation-preserving homeomorphism of a 3-cell is isotopic to the identity. Thus the Lemma is true. Suppose $g > 0$. Let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N . Since $h(\text{Im } P)$ is a regular neighborhood of v , a theorem in Section 1.5 supplies an isotopy, $H: N \times I \rightarrow N$, such that $H_1 \circ h(\text{Im } P) = \text{Im } P$. Therefore it can be assumed without loss of generality that $h(\text{Im } P) = \text{Im } P$. By Lemma 3.7.1 $h(P(v_i \times \{\epsilon\})) = P(v_i \times \{\epsilon\})$ for each $i = 1, \dots, g$ and $\epsilon = \pm 1$. Since h is orientation-preserving, $h|_{P(v_i \times \{\epsilon\})}: P(v_i \times \{\epsilon\}) \rightarrow P(v_i \times \{\epsilon\})$ is orientation-preserving for each $i = 1, \dots, g$ and $\epsilon = \pm 1$. It follows that there is an isotopy, $J: N \times I \rightarrow N$, which is invariant on $P(v_i \times \{\epsilon\})$ for each $i = 1, \dots, g$ and $\epsilon = \pm 1$ and such that $J_1 \circ h|_{P(v \times \{-1, 1\})}$ is the identity. Therefore it can be assumed without loss of generality that $h|_{P(v \times \{-1, 1\})}$ is the identity. Let $D(2g-1)$ denote the 2-cell with $2g-1$ holes, $(\text{Bd } N) \setminus P(v \times (-1, 1))$. Since $h|_{\text{Bd } D(2g-1)}$ is the identity, by Lemma 6.2.5, $h|_{D(2g-1)}$ is a twist of $D(2g-1)$ and $h|_{P((\text{Bd } v_i) \times [-1, 1])}$ is a twist of $P((\text{Bd } v_i) \times [-1, 1])$ for each $i = 1, \dots, g$. Suppose

$h|D(2g-1) = h_k \circ \dots \circ h_1$ where each h_i is a simple twist of $D(2g-1)$ or a homeomorphism which is mod $Bd D(2g-1)$ isotopic to the identity. If h_i is a simple twist of $D(2g-1)$ and A is an annulus in $D(2g-1)$ such that $h_i(x) = x$ for each $x \in D(2g-1) \setminus \text{Int } A$, let D be a properly embedded 2-cell in N and let $Q: D \times [-1,1] \rightarrow N$ be a 2-sided embedding of D in N such that $Q((Bd D) \times [-1,1]) = A$ and $\text{Im } Q \subset N \setminus P(v \times (-1,1))$. Then the cone construction on $\text{Im } Q$ supplies a simple twist of N , $T: N \rightarrow N$, such that $T|_{\text{Im } P}$ is the identity. On the other hand, if h_i is a homeomorphism of $D(2g-1)$ which is mod $Bd D(2g-1)$ isotopic to the identity, then h_i extends by the identity map to a homeomorphism h'_i of $Bd B$, where $B = N \setminus P(v \times (-1,1))$. Therefore h'_i is a homeomorphism of $Bd B$ which is mod $P(v \times \{-1,1\})$ isotopic to the identity. By a theorem in Section 1.3 h'_i extends to a homeomorphism h''_i of B which is mod $P(v \times \{-1,1\})$ isotopic to the identity. It follows that h_i extends to a twist of N , $T: N \rightarrow N$, such that $T|_{\text{Im } P}$ is the identity. Consequently, $h|D(2g-1)$ extends to a twist of N , $T_0: N \rightarrow N$, such that $T_0|_{\text{Im } P}$ is the identity. Since $h|P((Bd v_i) \times [-1,1])$ is a twist of $P((Bd v_i) \times [-1,1])$ for each $i = 1, \dots, g$ it follows in a similar fashion that $h|P((Bd v_i) \times [-1,1])$ extends to a twist of N , $T_i: N \rightarrow N$, such that $T_i|_{N \setminus P(v_i(-1,1))}$ is the identity. It follows that $T_g^{-1} \circ \dots \circ T_1^{-1} \circ T_0^{-1} \circ h|Bd N$ is the identity.

Furthermore, it may now be assumed without loss of generality that not only is $h|_{P(v \times [-1,1])}$ the identity but also $h|_{Bd N}$ is the identity. Since $h|_{Bd B}$ is the identity and $h|_{Bd P(v_i \times [-1,1])}$ is the identity for each $i=1, \dots, g$, it follows by a theorem in Section 1.3 that h is isotopic to the identity. Therefore the Lemma is proved. \square

Chapter 8: The proof of Lemma 2.2

Section 8.1: Band changes along compatible spanning arcs

Let N be a handlebody of genus $g \geq 2$, let v be a system of meridian disks in N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . If c, d are spanning arcs in the 2-cell with $2g-1$ holes,

$(\text{Bd } N) \setminus P(v \times (-1,1))$, and $(\text{Bd } c) \subset P(v_i \times \{\epsilon\}) \cup P(v_j \times \{\delta\})$ for some $i \neq j$, $\epsilon = \pm 1$, and $\delta = \pm 1$, then c and d are said to be compatible spanning arcs.

Lemma 8.1.1: Let N be a handlebody of genus $g \geq 2$, let v be a system of meridian disks for N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . Suppose c and d are compatible spanning arcs. If $v \mapsto v' = (v \cup v'_i) \setminus v_i$ is a band change along c and $v \mapsto v'' = (v \cup v''_i) \setminus v_i$ is a band change along d for some i , then there is a twist of N , $T: N \rightarrow N$, such that $T(v') = v''$.

Proof of Lemma 8.1.1: Let N be a handlebody of genus $g \geq 2$, let v be a system of meridian disks for N , and let $P: v \times [-1,1] \rightarrow N$ be a 2-sided embedding of v in N . Denote the 2-cell with $2g-1$ holes, $(\text{Bd } N) \setminus P(v \times (-1,1))$, by $D(2g-1)$. Suppose c and d are compatible spanning arcs in $D(2g-1)$. Suppose $v \mapsto v' = (v \cup v'_i) \setminus v_i$ is a band change along c and $v \mapsto v'' = (v \cup v''_i) \setminus v_i$ is a band change along d for some i . Let $H: N \times I \rightarrow N$ be a mod v

isotopy of N which is invariant on $P(v \times \{-1, 1\})$ and such that $H_1(Bd c) = Bd d$. Since $v \mapsto H_1(v')$ is a band change along $H_1(c)$, it can therefore be assumed without loss of generality that $Bd c = Bd d$. By Lemma 6.2.4 there is a twist h of $D(2g-1)$ such that $h(c) = d$. Let U be the regular neighborhood of $c \cup P((Bd v_i) \times \{\epsilon\}) \cup P((Bd v_j) \times \{\delta\})$ in $D(2g-1)$ where $Bd c \subset P(v_i \times \{\epsilon\}) \cup P(v_j \times \{\delta\})$ for some $j \neq i$, $\epsilon = \pm 1$, and $\delta = \pm 1$ and $Bd v_i'$ is the unique boundary component of U which lies in $\text{Int } D(2g-1)$. Since $h(c) = d$, $h(U)$ is a regular neighborhood of $d \cup P((Bd v_i) \times \{\epsilon\}) \cup P((Bd v_j) \times \{\delta\})$. It follows that there is a mod $Bd D(2g-1)$ isotopy, $H: D(2g-1) \times I \rightarrow D(2g-1)$, such that $H_1(h(Bd v_i')) = Bd v_i''$. Since h extends to a mod v twist of N and H extends to a mod v isotopy of N , it follows from a theorem in Section 1.8 that there is a twist T of N such that $T(v') = v''$. Thus the Lemma is proved. \square

Section 8.2: The proof of Lemma 2.2

Lemma 8.2.1: Let N be a handlebody of genus two and let v be a system of meridian disks for N . Suppose k is a system of curves in $Bd N$ such that no component of k is contractible in N . Suppose $h: N \rightarrow N$ is a homeomorphism such that k is cyclically reduced with respect to $h(v)$. If v' is a system of meridian disks for N and either $v \mapsto v'$ is a band change or there is an isotopy of N ,

$H: N \times I \rightarrow N$, such that $H_1(v) = v'$, then there is a twist of N , $T: N \rightarrow N$, such that k is cyclically reduced with respect to $T \circ h(v')$.

Proof of Lemma 8.2.1: Let N be a handlebody of genus two and let v be a system of meridian disks for N . Suppose k is a system of curves in $Bd N$ such that no component of k is contractible in N . Suppose $h: N \rightarrow N$ is a homeomorphism such that k is cyclically reduced with respect to $h(v)$. First suppose v' is a system of meridian disks for N and the substitution $v \mapsto v'$ is a band change. In fact, let $P: v \times [-1, 1] \rightarrow N$ be a 2-sided embedding of v in N and let the 2-cell with 3 holes, $(Bd N) \setminus P(v \times (-1, 1))$, be denoted by $D(3)$. Let c be a spanning arc in $D(3)$ with $Bd c \subset P(v_1 \times \{\epsilon\}) \cup P(v_{-1} \times \{\delta\})$ for some $\epsilon = \pm 1$ and $\delta = \pm 1$. Let U be a regular neighborhood of $c \cup P((Bd v_1) \times \{\epsilon\}) \cup P((Bd v_{-1}) \times \{\delta\})$ in $D(3)$ such that for some $i = \pm 1$, v'_i is a properly embedded 2-cell in $N \setminus P(v \times (-1, 1))$, $Bd v'_i$ is the unique boundary component of U which lies in $Int D(3)$, and $v' = (v \cup v'_i) \setminus v_i$. Define a 2-sided embedding of $h(v)$ in N as follows. If $(x, t) \in h(v) \times [-1, 1]$, let $P'(x, t) = h \circ P(h^{-1}(x), t)$. Let $D'(3)$ denote $(Bd N) \setminus P'(h(v) \times (-1, 1))$. Then $h(U)$ is a regular neighborhood of $h(c) \cup P'(Bd h(v_1) \times \{\epsilon\}) \cup P'(Bd h(v_{-1}) \times \{\delta\})$ in $D'(3)$. It follows that $h(v) \mapsto h(v')$ is a band change along the spanning arc $h(c)$. Since k is in general position

with $h(\text{Bd } v)$, there is a 2-sided embedding of $h(v)$ in N , $Q': h(v) \times [-1, 1] \rightarrow N$, such that k is in general position with Q' . By a theorem in Section 1.5 there is an isotopy, $J: N \times I \rightarrow N$, such that $J_1(\text{Im } P') = \text{Im } Q'$. The substitution $J_1 \circ h(v) \mapsto J_1 \circ h(v')$ is a band change along the spanning arc $J_1 \circ h(c)$ in the 2-cell with 3 holes,

$(\text{Bd } N) \setminus Q'(h(v) \times (-1, 1))$. Say

$\text{Bd}(J_1 \circ h(c)) \subset Q'(h(\text{Bd } v_1) \times \{\epsilon_1\}) \cup Q'(h(\text{Bd } v_{-1}) \times \{\delta_1\})$ for some $\epsilon_1 = \pm 1$ and $\delta_1 = \pm 1$. By Lemma 5.4.2 there is a spanning arc d in $(\text{Bd } N) \setminus Q'(h(v) \times (-1, 1))$ which is compatible with $J_1 \circ h(c)$ and a band change

$J \circ h(v) \mapsto w' = (J \circ h(v) \cup w'_i) \setminus J_1 \circ h(v_i)$ such that k is cyclically reduced with respect to w' . By Lemma 8.1.1 there is a twist of N , $T: N \rightarrow N$, such that $T(J_1 \circ h(v')) = w'$. Thus $T \circ J_1: N \rightarrow N$ is a twist of N and k is cyclically reduced with respect to $T \circ J_1 \circ h(v')$. Therefore $T \circ J_1$ is the desired twist of N . Suppose on the other hand that there is an isotopy of N , $H: N \times I \rightarrow N$, such that $H_1(v) = v'$. Then the twist of N desired by the Lemma is $h \circ H_1^{-1} \circ h^{-1}$, because $h \circ H_1^{-1} \circ h^{-1}$ is a homeomorphism of N which is isotopic to the identity and k is cyclically reduced with respect to $h \circ H_1^{-1} \circ h^{-1} \circ h(v')$. Thus the Lemma is proved. \square

Lemma 8.2.2: Let N be a handlebody of genus two and let v be a system of meridian disks for N . Suppose k is a system of curves in $\text{Bd } N$ and no component of k is

contractible in N . Then there is a twist of N , $T: N \rightarrow N$, such that k is cyclically reduced with respect to $T(v)$.

Proof of Lemma 8.2.2: Let N be a handlebody of genus two and let $v = v_1 \cup v_{-1}$ be a system of meridian disks for N . Suppose k is a system of curves in $Bd N$ and no component of k is contractible in N . Since there is an isotopy, $H: N \times I \rightarrow N$, such that k is in general position with $H_1(Bd v)$, it can be assumed without loss of generality that k is in general position with $Bd v$. By Corollary 4.1.2 there is a finite sequence $v, v', \dots, v^{(n)}$ with $n \geq 0$ of systems of meridian disks for N such that $v^{(i)} \mapsto v^{(i+1)}$ is a special T -transformation with respect to k for each $i = 0, \dots, n-1$ and k is cyclically reduced with respect to $v^{(n)}$. If $n = 0$, then the Lemma is true. Suppose $n > 0$. By Lemma 5.3.1 for each $i = 1, \dots, n$ the substitution $v^{(i)} \mapsto v^{(i-1)}$ is a band change or there is an isotopy of N , $H: N \times I \rightarrow N$, such that $H_1(v^{(i)}) = v^{(i-1)}$. In Lemma 8.2.1 let h be the identity ($=id$) homeomorphism and let $v^{(n)}$ be the given system of meridian disks. Then k is cyclically reduced with respect to $id(v^{(n)})$. By Lemma 8.2.1 there is a twist T_n of N such that k is cyclically reduced with respect to $T_n \circ id(v^{(n-1)})$. If $n-1 > 0$, in Lemma 8.2.1 let h be the homeomorphism $T_n \circ id$ and let $v^{(n-1)}$ be the given system of meridian disks. Then k is cyclically reduced with respect to $T_n \circ id(v^{(n-1)})$.

from the above. By Lemma 8.2.1 there is a twist T_{n-1} of N such that k is cyclically reduced with respect to $T_{n-1} \circ T_n \circ \text{id}(v^{(n-2)})$. Continuing in this manner, there are twists of N , T_1, \dots, T_n , such that k is cyclically reduced with respect to $T_1 \circ \dots \circ T_n(v)$. Thus the Lemma is proved. \square

Proof of Lemma 2.2: Let N be a handlebody of genus two and let v be a system of meridian disks for N . Suppose k is a system of curves in $\text{Bd} N$ and no component of k is contractible in N . By Lemma 8.2.2 there is a twist of N , $T: N \rightarrow N$, such that k is cyclically reduced with respect to $T(v)$. It follows that $T^{-1}(k)$ is cyclically reduced with respect to v . Since $T^{-1}: N \rightarrow N$ is a twist of N , T^{-1} is the desired twist of N . Thus the Lemma is proved. \square

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VITA

Robert J. Kramer was born in Quonset Point, Rhode Island, on April 20, 1952. He graduated from Lake Taylor High School in Norfolk, Virginia in June of 1969. In the spring of 1973 he received his Bachelor of Science degree from Virginia Polytechnic Institute, having majored in mathematics. In August of 1973 he began graduate study in the Department of Mathematics at Louisiana State University where he has been except for the academic year 1977-1978 which he spent as a graduate associate at the University of California, Los Angeles. He is now a candidate for a Doctor of Philosophy degree.

EXAMINATION AND THESIS REPORT

Candidate: Robert John Kramer

Major Field: Mathematics

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Approved:

B. G. Casler

Major Professor and Chairman

James B. Traynham

Dean of the Graduate School

EXAMINING COMMITTEE:

Theron S. Callen's

Kenneth E. Connerly

Lang Gides

O. Caruth McGehee

Date of Examination:

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